Chapter 1 Solutions

1.3 The probability density function for the gamma distribution is

\[ f(t \mid \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} \exp(-\lambda t). \]

What is the MTTF for the gamma distribution?

By definition,

\[ MTTF = \int_{-\infty}^{\infty} t f(t) \, dt \]

\[ = \int_{0}^{\infty} t \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} \exp(-\lambda t) \, dt \]

\[ = \int_{0}^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} t^\alpha \exp(-\lambda t) \, dt \]

Recognizing that this is a \( Gamma(\alpha + 1, \lambda) \) distribution, we can write the integral as

\[ \int_{0}^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} t^\alpha \exp(-\lambda t) \, dt = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha + 1)}{\lambda^{\alpha+1}} = \frac{\alpha}{\lambda}. \]

1.4 The probability density function for the Weibull distribution is

\[ f(t \mid \lambda, \beta, \theta) = \lambda \beta (t-\theta)^{\beta-1} \exp\left[-\lambda (t-\theta)^\beta\right], \quad 0 < \theta < t, \lambda > 0, \beta > 0. \]

What is the reliability function for the Weibull distribution?

By definition,

\[ R(t) = \int_{t}^{\infty} f(s) \, ds \]

\[ = \int_{t}^{\infty} \lambda \beta (s-\theta)^{\beta-1} \exp[-\lambda (s-\theta)^\beta] \, ds. \]

Noticing that if \( u = -\exp(-\lambda (s-\theta)^\beta) \), then \( du = \lambda \beta (s-\theta)^{\beta-1} \exp(-\lambda (s-\theta)^\beta) \, ds \), we have

\[ R(t) = -\exp(-\lambda (s-\theta)^\beta) \bigg|_{t}^{\infty} \]

\[ = \exp(-\lambda (t-\theta)^\beta). \]
1.5 The probability density function for an exponential random variable is
\[ f(t) = \lambda e^{-\lambda t}, \quad t > 0, \quad \lambda > 0. \]

What is the average hazard rate for an exponential random variable?

In Example 1.1.3, it is shown that the cumulative hazard function is
\[ H(t) = \lambda t. \]
Therefore, by definition, \( AHR(t_1, t_2) = \frac{\lambda(t_2-t_1)}{t_2-t_1} = \lambda. \)

1.7 Suppose that we are using the exponential distribution to model an item’s lifetime.

(a) We observe that the item failed at 6 hours. What is the likelihood function for this observation?

Let \( T \sim \text{Exponential}(\lambda). \) Then \( f(t \mid \lambda) = \lambda e^{-\lambda t} \) and the likelihood function is \( l(\lambda \mid t = 6) = \lambda e^{-6\lambda}. \)

(b) We observe that the item failed at some time between 5 and 10 hours. What is the likelihood function for this observation?

From Table 1.6, we know that the likelihood function for an interval censored observation has the form \( F(t_R) - F(t_L). \) The cumulative distribution function for an exponential random variable is \( F(t \mid \lambda) = 1 - \exp(-\lambda t). \) Consequently, the likelihood function is \( l(\lambda \mid 5 \leq t \leq 10) = \exp(-5\lambda) - \exp(-10\lambda). \)

(c) We observe the item for 20 hours, and it does not fail. What is the likelihood function for this observation?

From Table 1.6, we know that the likelihood function for a right-censored observation has the form \( 1 - F(t_R). \) The cumulative distribution function for an exponential random variable is \( F(t \mid \lambda) = 1 - \exp(-\lambda t). \) Consequently, the likelihood function is \( l(\lambda \mid t \geq 20) = 1 - \exp(-20\lambda). \)