SUPPLEMENT TO “POSTERIOR CONTRACTION AND CREDIBLE SETS FOR FILAMENTS OF REGRESSION FUNCTIONS”

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The supplementary file contains proofs for Lemma 5.7, 8.1, and 8.4.

Proof of Lemma 5.7. Let \( x_0 \) be an arbitrary point on \( \hat{\mathcal{L}} \). Thus \( \Upsilon_{x_0}(0) = x_0 \in \hat{\mathcal{L}} \) and \( \Upsilon_{x_0}(t_{x_0}) \in \mathcal{L} \) for some \( t_{x_0} > 0 \). Note that \( \inf_{y \in \mathcal{L}} \| x_0 - y \| \leq \| \Upsilon_{x_0}(0) - \Upsilon_{x_0}(t_{x_0}) \| \leq t_{x_0} \) as \( \| V(x) \| = 1 \). Let 
\[
D_{\Upsilon_{x_0},V} f(t) := \frac{d}{dt} f(\Upsilon_{x_0}(t)) = \nabla f(\Upsilon_{x_0}(t))^T V(\Upsilon_{x_0}(t))
\]
and 
\[
D^2_{\Upsilon_{x_0},V} f(t) := \langle \nabla \langle \nabla f(\Upsilon_{x_0}(t)), V(\Upsilon_{x_0}(t)), V(\Upsilon_{x_0}(t)) \rangle \nabla f(\Upsilon_{x_0}(t))^T V(\Upsilon_{x_0}(t)) + V(\Upsilon_{x_0}(t))^T H f(\Upsilon_{x_0}(t)) V(\Upsilon_{x_0}(t)),
\]
where the second line is due to the chain rule. A Taylor expansion yields that 
\[
D_{\Upsilon_{x_0},V} f(t) - D_{\Upsilon_{x_0},V} f(t_{x_0}) = (t - t_{x_0}) D^2_{\Upsilon_{x_0},V} f(\tilde{t})
\]
for some \( \tilde{t} \) between 0 and \( t_{x_0} \). In particular, since \( D_{\Upsilon_{x_0},V} f(t_{x_0}) = 0 \), letting \( t = 0 \), we obtain 
\[
D_{\Upsilon_{x_0},V} f(0) = -t_{x_0} D^2_{\Upsilon_{x_0},V} f(\tilde{t}).
\]
Furthermore, 
\[
|D_{\Upsilon_{x_0},V} f(0)| = |D_{\Upsilon_{x_0},V} f(0) - D_{\Upsilon_{x_0},V} \hat{f}(0)|
\]
\[
= |\nabla f(\Upsilon_{x_0}(0))^T V(\Upsilon_{x_0}(0)) - \nabla \hat{f}(\Upsilon_{x_0}(0))^T \hat{V}(\Upsilon_{x_0}(0))|
\]
\[
\leq \sup_x |\nabla f(x)^T V(x) - \nabla \hat{f}(x)^T \hat{V}(x)|
\]
\[
\leq \sup_x |\nabla f(x)^T (V(x) - \hat{V}(x))| + \sup_x |(\nabla f(x) - \nabla \hat{f}(x))^T \hat{V}(x))|
\]
\[
\leq C \sup_x \| V(x) - \hat{V}(x) \| + \sup_x \| \nabla f(x) - \nabla \hat{f}(x) \|
\]
\[
\leq C \sup_x \| V(x) - \hat{V}(x) \|.
\]
By the uniform continuity of \( \nabla f(x), \nabla V(x), V(x) \) and \( H f(x) \) and the continuity of \( \Upsilon_{x_0}(t) \) in \( t \), without loss of generality, we can make \( t_{x_0} \) small enough
The rate for sup $\|\nabla f(t)\|$ inequality this, if Assumption (A5) holds for the $x$ noting the highest degree of derivatives in each expression. We shall show small.

By Assumption (A3), $|D_{x_0}^2 V f(t_0)| > \eta$, and hence

$$|D_{x_0}^2 V f(t_0)| > \eta/2.$$ 

Therefore, $\inf_{y \in \mathcal{L}} \|x_0 - y\| \leq t_0 < \frac{C}{\eta} \sup_x \|V(x) - \hat{V}(x)\|$. Thus $d(\hat{L}|\mathcal{L}) \leq \frac{C}{\eta} \sup_x \|V(x) - \hat{V}(x)\|$. Similarly, $d(\mathcal{L}|\hat{L}) \leq \frac{C}{\eta} \sup_x \|V(x) - \hat{V}(x)\|$. Therefore, Haus($\mathcal{L}, \hat{L}$) $\leq \frac{C}{\eta} \sup_{x \in \{0,1\}^2} \|V(x) - \hat{V}(x)\|$. Recall that $V(x) = G(df^2(x))$.

Now since $G$ is a fixed Lipschitz continuous function, it is easy to get the upper bound for $\sup_{x \in \{0,1\}^2} \|V(x) - \hat{V}(x)\|$ in terms of the supremum distance of the derivatives of $f(x) - \hat{f}(x)$.

In the proof, $t_{x_0}$ can be made arbitrarily small in the limit. To see this, if Assumption (A5) holds for the $f$ (or more precisely, $\Upsilon_{x_0}$), then $\|\Upsilon_{x_0}(t_{x_0}) - \hat{\Upsilon}_{x_0}(t_{x_0})\| = \|\Upsilon_{x_0}(t_{x_0}) - x_0\| = \|\Upsilon_{x_0}(t_{x_0}) - \Upsilon_{x_0}(0)\| > C_d t_{x_0}$. Since $\|\Upsilon_{x_0}(t_{x_0}) - \hat{\Upsilon}_{x_0}(t_{x_0})\|$ can be made arbitrarily small due to the closedness in supremum norm (see previous theorems), $t_{x_0}$ can be made arbitrarily small.

**Proof of Lemma 8.1.** These results can be directly adapted from [1] by noting the highest degree of derivatives in each expression. We shall show the rates for $\sup_x \|H f(x) - H f^*(x)\|_F$, $\sup_x \|\nabla d^2 f(x) - \nabla d^2 f^*(x)\|_F$, and $\sup_x \|\nabla V(x) - \nabla V^*(x)\|_F$. Notice that

$$\|H f(x) - H f^*(x)\|_F = \left( |f(2,0) - f^*(2,0)|^2 + 2|f(1,1) - f^*(1,1)|^2 \right)^{1/2},$$

$$\leq 2^{-1/2} \left( |f(2,0) - f^*(2,0)| + 2|f(1,1) - f^*(1,1)| \right).$$

The rate for $\sup_x \|H f(x) - H f^*(x)\|_F$ then follows easily.

Also, the contraction rates for $\|\nabla d^2 f(x) - \nabla d^2 f^*(x)\|_F$ follows from the inequality $\|\nabla d^2 f(x) - \nabla d^2 f^*(x)\|_F \leq 6^{-1/2} \sum_{r:|r|=3} |D^r f(x) - D^r f^*(x)|$. 
Since $\nabla V(x) - \nabla V^*(x) = \nabla G(d^2 f(x))\nabla d^2 f(x) - \nabla G(d^2 f^*(x))\nabla d^2 f^*(x)$, which is 2 by 2 matrix, straightforward calculation gives its $(1,1)$ element
\[
\begin{align*}
&\left(G_1^{(1,0,0)}(d^2 f(x))f^{(3,0)}(x) - G_1^{(1,0,0)}(d^2 f^*(x))f^{*(3,0)}(x)\right), \\
&+ \left(G_1^{(0,1,0)}(d^2 f(x))f^{(2,1)}(x) - G_1^{(0,1,0)}(d^2 f^*(x))f^{*(2,1)}(x)\right), \\
&+ \left(G_1^{(0,0,1)}(d^2 f(x))f^{(1,2)}(x) - G_1^{(0,0,1)}(d^2 f^*(x))f^{*(1,2)}(x)\right).
\end{align*}
\]

The absolute value of the first summand is bounded by the sum of
\[
\left|G_1^{(1,0,0)}(d^2 f(x))f^{(3,0)}(x) - G_1^{(1,0,0)}(d^2 f^*(x))f^{*(3,0)}(x)\right|
\]
and
\[
\left|G_1^{(1,0,0)}(d^2 f(x))f^{*(3,0)}(x) - G_1^{(1,0,0)}(d^2 f^*(x))f^{*(3,0)}(x)\right|.
\]

Noting that $d^2 f(x)$ contracts to $d^2 f^*(x)$ uniformly in $x$, hence $\{d^2 f(x) : x \in [0,1]^2\} \subset Q_8$ with posterior probability tending to 1, the first term is bounded by a constant multiple of $|f^{(3,0)}(x) - f^{*(3,0)}(x)|$, in view of the result 4 of Remark 8.1. By the same remark and assumption $\|f^*\|_{0, \infty} < \infty$, the second term is bounded by $\|d^2 f(x) - d^2 f^*(x)\||f^{(3,0)}(x)|$. Using similar arguments for the second and third summand, one can see that the absolute value of the $(1,1)$ element of $\nabla V(x) - \nabla V^*(x)$ is bounded by $|f^{(3,0)}(x) - f^{*(3,0)}(x)| + |f^{(2,1)}(x) - f^{*(2,1)}(x)| + |f^{(1,2)}(x) - f^{*(1,2)}(x)|$.

Dealing the rest elements of $\nabla V(x) - \nabla V^*(x)$ similarly, we can see that $\|\nabla V(x) - \nabla V^*(x)\|_F \lesssim \sum_{|r|=3} |D^r f - D^r f^*|$. \hfill \Box

**Proof of Lemma 8.4.** We shall consider only two separate cases (i) $r = (2, 0)$ and (ii) $r = (1, 1)$.

(i). For $r = (2, 0)$ (similarly for $r = (0, 2)$),
\[
\|a(x_0, t)\|_1 = \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left|\int_0^t \tilde{G}_1(s)B_{j_1}''(\Upsilon_{1,x_0}^*(s))B_{j_2}(\Upsilon_{2,x_0}^*(s))ds\right|
\lesssim \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \int_0^t |B_{j_1}''(\Upsilon_{1,x_0}^*(s))|B_{j_2}(\Upsilon_{2,x_0}^*(s))ds
\leq \sum_{j_1}^{J_1} \int_0^t |B_{j_1}''(\Upsilon_{1,x_0}^*(s))|ds,
\]
the last line follows by \( \sum_{j_2 = 1} B'_{j_2}(x_2) = 1 \) and noticing that

\[
B''_{j_1,q_1}(x) = \frac{(q_1 - 1)(q_1 - 2) B_{j_1,q_1-2}(x)}{(t_{j_1} - t_{j_1-1})(t_{j_1} - t_{j_1-2})} + \frac{(q_1 - 1)(q_1 - 2) B_{j_1-1,q_1-2}(x)}{(t_{j_1-1} - t_{j_1-2})(t_{j_1-1} - t_{j_1-3})},
\]

which implies \( \sum_{j_1 = 1} B''_{j_1} \mid \mathcal{Y}_{x_0}(s) \mid \leq 4J^2 \). Therefore, \( \|a(x_0,t)\|_1 \lesssim J^2 \).

Next, let \( S_{j_1} = [t_{1,j_1 - (q_1 - 2)}, t_{1,j_1}], S_{j_2} = [t_{2,j_2 - q_2}, t_{2,j_2}] \) and \( 1_{j_1,j_2}(s) := 1 \{ s : \mathcal{Y}_{x_0}(s) \in S_{j_1} \times S_{j_2} \} \). Turn to \( \|a(x_0,t)\|_2^2 \), which is

\[
\sum_{j_1 = 1}^J \sum_{j_2 = 1}^J \left( \int_0^t \bar{G}_1(s) B''_{j_1} \mid \mathcal{Y}_{1,x_0}(s) \mid B_{j_2} \mid \mathcal{Y}_{2,x_0}(s) \right) ds \leq 4J^2.
\]

The last equality is obtained as follows. Since for any fix \( j_1, j_2, B''_{j_1} \cdot \) is supported on \( S_{j_1} \) and \( B_{j_2}(\cdot) \) is supported on \( S_{j_2} \). So \( 1_{j_1,j_2}(s) \) is supported on \( S_{j_1} \times S_{j_2} \) and \( \mathcal{Y}_{x_0}(s) \in S_{j_1} \times S_{j_2} \).

Notice that for \( n \) large, \( |S_{j_1}| \times |S_{j_2}| \lesssim J^{-1} \) and \( \mathcal{Y}_{x_0}(s) \in S_{j_1} \times S_{j_2} \). Notice that for \( n \) large, \( |S_{j_1}| \times |S_{j_2}| \lesssim J^{-1} \) and \( \mathcal{Y}_{x_0}(s) \in S_{j_1} \times S_{j_2} \).

By Assumption (A5), \( C_0 |s - s'| \leq |s - s'| \) implies that \( \| \mathcal{Y}_{x_0}(s) - \mathcal{Y}_{x_0}(s') \| \lesssim J^{-1} \), and hence \( |s - s'| \lesssim J^{-1} \). Therefore, for \( n \) large enough, above quantity can be further bounded by a constant multiple of

\[
\int_0^t \int_0^t \sum_{j_1 = 1}^J \sum_{j_2 = 1}^J 1 \{ |s - s'| < CJ^{-1} \} \left( |B''_{j_1} \mid \mathcal{Y}_{1,x_0}(s) \mid B_{j_2} \mid \mathcal{Y}_{2,x_0}(s) \right) ds ds'.
\]
Noting \( B_{j_2}(x_2) \leq 1 \) and \( \sum_{j_2=1}^{J_2} B_{j_2} = 1 \), it can be further bounded by

\[
\begin{align*}
\int_0^t \int_0^t \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} 1 \{ |s - s'| < CJ^{-1} \} & \left( |B''_{j_1}(\Upsilon_{1,x_0}^*(s))| B_{j_2}(\Upsilon_{2,x_0}^*(s)) \right) \\
& \times |B''_{j_1}(\Upsilon_{1,x_0}^*(s'))| \right) dsds' \\
& \lesssim \int_0^t \int_0^t \sum_{j_1=1}^{J_1} 1 \{ |s - s'| < CJ^{-1} \} \left( |B''_{j_1}(\Upsilon_{1,x_0}^*(s))| B''_{j_1}(\Upsilon_{1,x_0}^*(s')) \right) dsds' \\
& \lesssim J^2 J^{-1} \int_0^t \sum_{j_1=1}^{J_1} |B''_{j_1}(\Upsilon_{1,x_0}^*(s))| ds.
\end{align*}
\]

From argument used in bounding \( \|a(x_0,t)\|_1 \), we have \( \sum_{j_1=1}^{J_1} |B''_{j_1}(\Upsilon_{1,x_0}^*)| \lesssim J^2 \). This completes the proof for \( \|a(x_0,t)\|_2 \lesssim J^3 \).

For the third result, we write

\[
\|a(x_0,t) - a(\tilde{x}_0,\tilde{t})\|^2 \lesssim \|a_1(x_0,t,\tilde{t})\|^2 + \|a_2(\tilde{t},x_0,\tilde{x}_0)\|^2,
\]

where

\[
a_1(x_0,t,\tilde{t}) := \int_0^t \tilde{G}(\Upsilon_{x_0}^*(s)) b_{j_1,j_2}(\Upsilon_{x_0}^*(s)) ds - \int_0^{\tilde{t}} \tilde{G}(\Upsilon_{x_0}^*(s)) b_{j_1,j_2}(\Upsilon_{x_0}^*(s)) ds,
\]

and

\[
a_2(\tilde{t},x_0,\tilde{x}_0) := \int_0^{\tilde{t}} \tilde{G}(\Upsilon_{x_0}^*(s)) b_{j_1,j_2}(\Upsilon_{x_0}^*(s)) ds - \int_0^{\tilde{t}} \tilde{G}(\Upsilon_{\tilde{x}_0}^*(s)) b_{j_1,j_2}(\Upsilon_{\tilde{x}_0}^*(s)) ds.
\]

First, note that

\[
\|a_1(x_0,t,\tilde{t})\|^2 = \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^t \tilde{G}(\Upsilon_{x_0}^*(s)) B''_{j_1}(\Upsilon_{1,x_0}^*(s)) B_{j_2}(\Upsilon_{2,x_0}^*(s)) ds \right)^2 \lesssim J^2 J^{-1} \int_0^{\tilde{t}} (4J^2) ds = J^3 |t - \tilde{t}|,
\]

where the second line follows by a similar argument used to bound \( \|a(x_0,t)\|^2 \). Next, \( \|a_2(\tilde{t},x_0,\tilde{x}_0)\|^2 \) is given by

\[
\sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left[ \int_0^{\tilde{t}} \left( \tilde{G}(\Upsilon_{x_0}^*(s)) B''_{j_1}(\Upsilon_{1,x_0}^*(s)) B_{j_2}(\Upsilon_{2,x_0}^*(s)) \right) \right. \\
\left. - \tilde{G}(\Upsilon_{\tilde{x}_0}^*(s)) B''_{j_1}(\Upsilon_{1,x_0}^*(s)) B_{j_2}(\Upsilon_{2,x_0}^*(s)) ds \right]^2,
\]
which is bounded (up to a multiple constant) by

\[
\sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^\ell \tilde{G}(\Upsilon_{x_0}(s)) \left( B''_{j_2} (\Upsilon_{1,x_0}(s)) - B''_{j_1} (\Upsilon_{1,\tilde{x}_0}(s)) \right) B_{j_2} (\Upsilon_{2,x_0}(s)) ds \right)^2 \\
+ \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^\ell \tilde{G}(\Upsilon_{x_0}(s)) \left( B_{j_2} (\Upsilon_{2,x_0}(s)) - B_{j_2} (\Upsilon_{2,\tilde{x}_0}(s)) \right) B''_{j_1} (\Upsilon_{1,x_0}(s)) ds \right)^2 \\
+ \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^\ell \tilde{G}(\Upsilon_{x_0}(s)) - \tilde{G}(\Upsilon_{\tilde{x}_0}(s)) \right) B''_{j_1} (\Upsilon_{1,x_0}(s)) B_{j_2} (\Upsilon_{2,x_0}(s)) ds \right)^2.
\]

Bound the first term in the right hand side of above expression as

\[
\sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^\ell \tilde{G}(\Upsilon_{x_0}(s)) \left( B''_{j_1} (\Upsilon_{1,x_0}(s)) - B''_{j_1} (\Upsilon_{1,\tilde{x}_0}(s)) \right) B_{j_2} (\Upsilon_{2,x_0}(s)) ds \right)^2 \\
\lesssim \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^\ell |\tilde{G}(\Upsilon_{x_0}(s))||B''_{j_1} (\Upsilon_{1,x_0}(s))||\Upsilon_{1,x_0}(s) - \Upsilon_{1,\tilde{x}_0}(s)| \\
\times B_{j_2} (\Upsilon_{2,x_0}(s)) ds \right)^2 \\
\lesssim \|x_0 - \tilde{x}_0\|^2 \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^\ell |\tilde{G}(\Upsilon_{x_0}(s))||B''_{j_1} (\Upsilon_{1,x_0}(s))| B_{j_2} (\Upsilon_{2,x_0}(s)) ds \right)^2 \\
\lesssim \|x_0 - \tilde{x}_0\|^2 J^2 \int_0^\ell |B''_{j_1} (\Upsilon_{1,x_0}(s))| ds \\
\lesssim \|x_0 - \tilde{x}_0\|^2 J^5.
\]

The second term is bounded as

\[
\sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^\ell \tilde{G}(\Upsilon_{x_0}(s)) \left( B_{j_2} (\Upsilon_{2,x_0}(s)) - B_{j_2} (\Upsilon_{2,\tilde{x}_0}(s)) \right) B''_{j_1} (\Upsilon_{1,x_0}(s)) ds \right)^2 \\
\lesssim \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^\ell |\tilde{G}(\Upsilon_{x_0}(s))||\Upsilon_{2,x_0}(s) - \Upsilon_{2,\tilde{x}_0}(s)||B'_{j_2} (\Upsilon_{2,x_0}(s))| |B''_{j_1} (\Upsilon_{1,x_0}(s))| \right)^2 \\
\lesssim \|x - \tilde{x}_0\|^2 \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^\ell |\tilde{G}(\Upsilon_{x_0}(s))||B'_{j_2} (\Upsilon_{2,x_0}(s))||B''_{j_1} (\Upsilon_{1,x_0}(s))| \right)^2
\]
\begin{equation*}
\lesssim \|x - \tilde{x}_0\|^2 \int_0^t \mathbb{1}\{ |s - s'| \leq CJ^{-1} \} \times \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} |B'_{j_2}(\Upsilon^*_{2, \tilde{x}_0}(s))| |B''_{j_1}(\Upsilon^*_{1, \tilde{x}_0}(s))| |B''_{j_2}(\Upsilon^*_{2, \tilde{x}_0}(s'))| dsds' \\
\lesssim \|x - \tilde{x}_0\|^2 \int_0^t \mathbb{1}\{ |s - s'| \leq CJ^{-1} \} J^2 \sum_{j_1=1}^{J_1} |B''_{j_1}(\Upsilon^*_{1, \tilde{x}_0}(s))| |B''_{j_2}(\Upsilon^*_{2, \tilde{x}_0}(s'))| dsds' \\
\lesssim J^5 \|x_0 - \tilde{x}_0\|^2,
\end{equation*}

where the second line follows from the mean value theorem, the third line from the Lipschitz continuity of $\Upsilon^*_{x_0}$ in $x_0$ (Remark 8.1) and fourth line by a similar argument used to bound $\|a(x, t)\|^2$. The third term

\begin{equation*}
\sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^t \left( \tilde{G}(\Upsilon^*_{x_0}(s)) - \tilde{G}(\Upsilon^*_{\tilde{x}_0}(s)) \right) B''_{j_1}(\Upsilon^*_{1, \tilde{x}_0}(s)) B_{j_2}(\Upsilon^*_{2, \tilde{x}_0}(s)) ds \right)^2 \\
\lesssim \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \int_0^t \left( \|\Upsilon^*_{x_0}(s) - \Upsilon^*_{\tilde{x}_0}(s)\|^2 B''_{j_1}(\Upsilon^*_{1, \tilde{x}_0}(s)) B_{j_2}(\Upsilon^*_{2, \tilde{x}_0}(s)) ds \right)^2 \\
\lesssim \|x_0 - \tilde{x}_0\|^2 J^3,
\end{equation*}

where the second line holds by mean value theorem and the third line holds due to Lipschitz continuity of $\Upsilon^*_{x_0}$ in $x_0$ (Remark 8.1) and last line holds by similar argument for $\|a(x_0, t)\|^2$.

In summary, we have that $\|a_2(\tilde{t}, x_0, \tilde{x}_0)\| \lesssim J^5 \|x_0 - \tilde{x}_0\|^2$ and $\|a_1(x_0, t, \tilde{t})\| \lesssim J^3 |t - \tilde{t}|$.

(ii). Now turning to the case $r = (1, 1)$. By similar argument we have

\begin{equation*}
\|a(x_0, t)\|_1 = \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left| \int_0^t \tilde{G}_1(s) B'_{j_1}(\Upsilon^*_{1, x_0}(s)) B'_{j_2}(\Upsilon^*_{2, x_0}(s)) ds \right| \\
\lesssim \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \int_0^t |B'_{j_1}(\Upsilon^*_{1, x_0}(s))| |B'_{j_2}(\Upsilon^*_{2, x_0}(s))| ds \\
\leq J \sum_{j_1=1}^{J_1} \int_0^t |B'_{j_1}(\Upsilon^*_{1, x_0}(s))| ds \lesssim J^2
\end{equation*}
Likewise, \( \|a(x_0, t)\|^2 = \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^t \tilde{G}(s) B_{j_1}^t (Y_{1,x_0}^* (s)) B_{j_2}^t (Y_{2,x_0}^* (s)) ds \right)^2 \) can be bounded by

\[
\int_0^t \int_0^t \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} 1 \{|s-s'| < C J^{-1}\} \left( |B_{j_1}^t (Y_{1,x_0}^* (s))| B_{j_2}^t (Y_{2,x_0}^* (s)) \right) ds ds' \leq J \int_0^t \int_0^t \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} 1 \{|s-s'| < C J^{-1}\} \left( |B_{j_1}^t (Y_{1,x_0}^* (s))| |B_{j_2}^t (Y_{2,x_0}^* (s))| \right) ds ds' \leq J^2 \int_0^t \int_0^t \sum_{j_1=1}^{J_1} 1 \{|s-s'| < C J^{-1}\} \left( |B_{j_1}^t (Y_{1,x_0}^* (s))| \right) ds ds' \leq J^3 J^{-1} \int_0^t \sum_{j_1=1}^{J_1} |B_{j_1}^t (Y_{1,x_0}^* (s))| ds
\]

which is bounded by a constant multiple of \( J^3 \).

The third result \( \|a_1(x_0, t, \tilde{t})\|^2 \leq J^3 |t-\tilde{t}| \) and \( \|a_2(\tilde{t}, x, \tilde{x}_0)\|^2 \leq J^5 \|x_0 - \tilde{x}_0\|^2 \) can be derived in a similar manner, since

\[
\|a_1(x_0, t, \tilde{t})\|^2 = \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^{\tilde{t}} \tilde{G}(Y_{x_0}^* (s)) B_{j_1}^t (Y_{1,x_0}^* (s)) B_{j_2}^t (Y_{2,x_0}^* (s)) ds \right)^2 \leq J^2 J^{-1} \int_0^{\tilde{t}} (4J^2) ds
\]

which equals to \( J^3 |t-\tilde{t}| \); where the second line follows by a similar argument used in bounding \( \|a(x_0, t)\|^2 \).

Next, to bound \( \|a_2(\tilde{t}, x_0, \tilde{x}_0)\|^2 \), we need to estimate the following three terms

\[
\sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^{\tilde{t}} \tilde{G}(Y_{x_0}^* (s)) \left( B_{j_1}^t (Y_{1,x_0}^* (s)) - B_{j_1}^t (Y_{1,\tilde{x}_0}^* (s)) \right) B_{j_2}^t (Y_{2,x_0}^* (s)) ds \right)^2,
\]
\[ \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^\tilde{t} \tilde{G}(\tilde{Y}_{x_0}(s)) \left( B'_{j_2} (\tilde{Y}_{2,x_0}^*(s)) - B'_{j_2} (\tilde{Y}_{2,\tilde{x}_0}^*(s)) \right) B'_{j_1} (Y_{1,\tilde{x}_0}^*(s)) ds \right)^2, \]
\[ \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left( \int_0^\tilde{t} \left( \tilde{G}(Y_{x_0}^*(s)) - \tilde{G}(Y_{\tilde{x}_0}^*(s)) \right) B'_{j_1} (Y_{1,\tilde{x}_0}^*(s)) B'_{j_2} (Y_{2,\tilde{x}_0}^*(s)) ds \right)^2. \]

This can be done by similar argument used for the case \( r = (2,0) \) and we omit the details.

References.