(7) Bayesian linear regression

ST440/540: Applied Bayesian Statistics

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Bayesian linear regression

- Linear regression is by far the most common statistical model
- It includes as special cases the t-test and ANOVA
- The multiple linear regression model is

\[ Y_i \sim \text{Normal}(\beta_0 + X_{i1}\beta_1 + \ldots + X_{ip}\beta_p, \sigma^2) \]

independently across the \( i = 1, \ldots, n \) observations

- As we’ll see, Bayesian and classical linear regression are similar if \( n \gg p \) and the priors are uninformative.
- However, the results can be different for challenging problems, and the interpretation is different in all cases.
Review of least squares

- The least squares estimate of \( \beta = (\beta_0, \beta_1, ..., \beta_p)^T \) is

\[
\hat{\beta}_{OLS} = \arg\min_\beta \sum_{i=1}^{n} (Y_i - \mu_i)^2
\]

where \( \mu_i = \beta_0 + X_{i1}\beta_1 + ... + X_{ip}\beta_p \)

- \( \hat{\beta}_{OLS} \) is unbiased even if the errors are non-Gaussian

- If the errors are Gaussian then the likelihood is proportional to

\[
\prod_{i=1}^{n} \exp \left[ -\frac{(Y_i - \mu_i)^2}{2\sigma^2} \right] = \exp \left[ -\frac{\sum_{i=1}^{n}(Y_i - \mu_i)^2}{2\sigma^2} \right]
\]

- Therefore, if the errors are Gaussian \( \hat{\beta}_{OLS} \) is also the MLE
Review of least squares

- Linear regression is often simpler to describe using linear algebra notation

- Let $\mathbf{Y} = (Y_1, ..., Y_n)^T$ be the response vector and $\mathbf{X}$ be the $n \times (p + 1)$ matrix of covariates

- Then the mean of $\mathbf{Y}$ is $\mathbf{X}\beta$ and the least squares solution is

$$\hat{\beta}_{OLS} = \underset{\beta}{\mathrm{argmin}} (\mathbf{Y} - \mathbf{X}\beta)^T(\mathbf{Y} - \mathbf{X}\beta) = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$$

- If the errors are Gaussian then the sampling distribution is

$$\hat{\beta}_{OLS} \sim \text{Normal} \left[ \beta, \sigma^2(\mathbf{X}^T\mathbf{X})^{-1} \right]$$

- If the variance $\sigma^2$ is estimated using the mean squared residual error then the sampling distribution is multivariate t
Bayesian regression

- The likelihood remains

\[ Y_i \sim \text{Normal}(\beta_0 + X_{i1}\beta_1 + \ldots + X_{ip}\beta_p, \sigma^2) \]

independent for \( i = 1, \ldots, n \) observations

- As with a least squares analysis, it is crucial to verify this is appropriate using qq-plots, added variable plots, etc.

- A Bayesian analysis also requires priors for \( \beta \) and \( \sigma \)

- We will focus on prior specification since this piece is uniquely Bayesian.
For the purpose of setting priors, it is helpful to standardize both the response and each covariate to have mean zero and variance one.

Many priors for $\beta$ have been considered:

1. Improper priors
2. Gaussian priors
3. Double exponential priors
4. Many, many more...
Improper priors

- The Jeffreys’ prior is flat $p(\beta) = 1$

- This is improper, but the posterior is proper under the same conditions required by least squares

- If $\sigma$ is known then

$$\beta | Y \sim \text{Normal} \left[ \hat{\beta}_{OLS}, \sigma^2 (X^TX)^{-1} \right]$$


- Therefore, the results should be similar to least squares

- How are they different?
Improper priors

- Of course we rarely know $\sigma$

- Typically the error variance follows an $\text{InvGamma}(a, b)$ prior with $a$ and $b$ set to be small, say $a = b = 0.01$.

- In this case the posterior of $\beta$ follows a multivariate $t$ centered on $\hat{\beta}_{OLS}$

- Again, the results are similar to OLS

- The objective Bayes Jeffreys prior for $\theta = (\beta, \sigma)$ is

$$p(\beta, \sigma^2) = \frac{1}{\sigma^2}$$

which is the limit as $a, b \to 0$
Another common prior for $\beta$ is Zellner’s g-prior

$$\beta \sim \text{Normal} \left[ 0, \frac{\sigma^2}{g} (X^T X)^{-1} \right]$$

This prior is proper assuming $X$ is full rank.

The posterior mean is

$$\frac{1}{1 + g\hat{\beta}_{OLS}}$$

This shrinks the least estimate towards zero.

$g$ controls the amount of shrinkage.

$g = 1/n$ is common, and called the unit information prior.
Univariate Gaussian priors

- If there are many covariates or the covariates are collinear, then $\hat{\bm{\beta}}_{OLS}$ is unstable.

- Independent priors can counteract collinearity

$$\beta_j \sim \text{Normal}(0, \sigma^2/g)$$

independent over $j$

- The posterior mode is

$$\arg\min_{\bm{\beta}} \sum_{i=1}^{n} (Y_i - \mu_i)^2 + g \sum_{j=1}^{p} \beta_j^2$$

- In classical statistics, this is known as the ridge regression solution and is used to stabilize the least squares solution.
An increasingly-popular prior is the double exponential or Bayesian LASSO prior. The prior is \( \beta_j \sim \text{DE}(\tau) \) which has PDF

\[
f(\beta) \propto \exp \left( -\frac{|\beta|}{\tau} \right)
\]

The square in the Gaussian prior is replaced with an absolute value. The shape of the PDF is thus more peaked at zero (next slide). The BLASSO prior favors settings where there are many \( \beta_j \) near zero and a few large \( \beta_j \). That is, \( \rho \) is large but most of the covariates are noise.
BLASSO

Prior

Gaussian

BLASSO

β
The posterior mode is

\[
\arg\min_{\beta} \sum_{i=1}^{n} (Y_i - \mu_i)^2 + g \sum_{j=1}^{p} |\beta_j|
\]

In classical statistics, this is known as the LASSO solution.

It is popular because it adds stability by shrinking estimates towards zero, and also sets some coefficients to zero.

Covariates with coefficients set to zero can be removed.

Therefore, LASSO performs variables selection and estimation simultaneously.
With flat or Gaussian (with fixed prior variance) priors the posterior is available in closed-form and Monte Carlo sampling is not needed.

With normal priors all full conditionals are Gaussian or inverse gamma, and so Gibbs sampling is simple and fast.

JAGS works well, but there are R (and SAS and others) packages dedicated just to Bayesian linear regression that are preferred for big/hard problems.

BLR is probably the most common.

http://www4.stat.ncsu.edu/~reich/ABA/code/regJAGS
Computing for the BLASSO

- For the BLASSO prior the full conditionals are more complicated
- There is a trick to make all full conditional conjugate so that Gibbs sampling can be used
- Metropolis sampling works fine too
- `BLR` works well for BLASSO and is super fast
- `JAGS` can handle this as well,
- `http://www4.stat.ncsu.edu/~reich/ABA/code/BLASSO`
Summarizing the results

► The standard summary is a table with marginal means and 95% intervals for each $\beta_j$

► This becomes unwieldy for large $p$

► Picking a subset of covariates is a crucial step in a linear regression analysis.

► We will discuss this later in the course.

► Common methods include cross-validation, information criteria, and stochastic search.
Logistic regression

- Other forms of regression follow naturally from linear regression
- For example, for binary responses $Y_i \in \{0, 1\}$ we might use logistic regression

$$
\text{logit}[\text{Prob}(Y_i = 1)] = \eta_i = \beta_0 + \beta_1 X_{i1} + \ldots + \beta_p X_{ip}
$$

- The logit link is the log-odd $\text{logit}(x) = \log[x/(1 - x)]$
- Then $\beta_j$ represents the increase in the log odds of an event corresponding to a one-unit increase in covariate $j$
- The expit transformation $\text{expit}(x) = \exp(x)/[1 + \exp(x)]$ is the inverse, and

$$
\text{Prob}(Y_i = 1) = \text{expit}(\eta_i) \in [0, 1]
$$
Logistic regression

- Bayesian logistic regression requires a prior for $\beta$

- All of the prior we have discussed for linear regression (Zellner, BLASSO, etc) apply

- Computationally the full conditional distributions are no longer conjugate and so we must use Metropolis sampling

- The R function `MCMClogit` does this efficiently

- It is fast in JAGS too, for example [http://www4.stat.ncsu.edu/~reich/ABA/code/GLM](http://www4.stat.ncsu.edu/~reich/ABA/code/GLM)
Predictions

Say we have a new covariate vector $\mathbf{X}_{new}$ and we would like to predict the corresponding response $Y_{new}$.

A plug-in approach would fix $\beta$ and $\sigma$ at their posterior means $\hat{\beta}$ and $\hat{\sigma}$ to make predictions

$$Y_{new | \hat{\beta}, \hat{\sigma} \sim \text{Normal}(\mathbf{X}_{new} \hat{\beta}, \hat{\sigma}^2)}$$

However this plug-in approach suppresses uncertainty about $\beta$ and $\sigma$.

Therefore these prediction intervals will be slightly too narrow leading to undercoverage.
Posterior predictive distribution (PPD)

- We should really account for all uncertainty when making predictions, including our uncertainty about $\beta$ and $\sigma$

- We really want the PPD

$$p(Y_{new} | Y) = \int f(Y_{new}, \beta, \sigma | Y) \, d\beta \, d\sigma$$  

$$= \int f(Y_{new} | \beta, \sigma) f(\beta, \sigma | Y) \, d\beta \, d\sigma$$

- Marginalizing over the model parameters accounts for their uncertainty

- The concept of the PPD applies generally (e.g., logistic regression) and means the distribution of the predicted value marginally over model parameters
Posterior predictive distribution (PPD)

- MCMC naturally gives draws from $Y_{new}$’s PPD
  - For MCMC iteration $t$ we have $\beta(t)$ and $\sigma(t)$
  - For MCMC iteration $t$ we sample
    \[ Y_{new}^{(t)} \sim \text{Normal}(X\beta(t), \sigma(t)^2) \]
  - $Y_{new}^{(1)}, \ldots, Y_{new}^{(S)}$ are samples from the PPD

- This is an example of the claim that “Bayesian methods naturally quantify uncertainty”

- [http://www4.stat.ncsu.edu/~reich/ABA/code/Predict](http://www4.stat.ncsu.edu/~reich/ABA/code/Predict)