because of similarity. The parameter h for a given object can be thought of as the radius of a sphere that has the same ratio of V to A as the object. It will henceforth be referred to as the object's *harmonic parameter*.

To check how $h = nV/A$ works on some examples, consider the three simple cases of FIGURE 1, plus that of a right circular cone.

Circles: radius r, $h = 2(\pi r^2)/(2\pi r) = r$.

Squares: side $2r$, $h = 2(2r)^2/4(2r) = r$.

Rectangles: length $3r$, width $2r$, $h = 2(3r2r)/2(3r + 2r) = 6r/5$.

Cones: radius r, height ar,

$$h = 3\left(\frac{1}{3}\pi r^2 ar\right) \Big/ (\pi r^2 + \pi r\sqrt{r^2 + (ar)^2}) = ar/(1 + \sqrt{1 + a^2}).$$

If you can assign a volume and an area to your coffee cup, it will have a value for its harmonic parameter h. The magnitude of h is somewhat indicative of function. For a lung and a balloon of the same volume, the lung will have a much smaller h than the balloon, and the same holds in comparing a brain and a stomach. Returning to the vernacular, one might say that increasing superficiality provides for higher order functioning.

As might be expected, the generalization invoked by the idea of the harmonic parameter can be extended to higher derivatives and also to multi-parameter objects. It must be confessed that the authors found it quite entertaining to do so, for the harmonic mean kept showing up in unexpected ways, which they believe readers might now enjoy discovering by themselves.

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Markov Chains for the *RISK* Board Game Revisited

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Probabilistic reasoning goes a long way in many popular board games. Abbott and Richey [**1**] and Ash and Bishop [**2**] identify the most profitable properties in *Monopoly*, and Tan [**3**] derives battle strategies for *RISK*. In *RISK*, the stochastic progress of a battle between two players over any of the 42 countries can be described using a Markov chain. Theory of Markov chains can be applied to address questions about the probabilities of victory and expected losses in battle.

Tan addresses two interesting questions:

> *If you attack a territory with your armies, what is the probability that you will capture this territory? If you engage in a war, how many armies should you expect to lose depending on the number of armies your opponent has on that territory?* [**3**, p. 349]

A mistaken assumption of independence leads to a slight misspecification of the transition probability matrix for the system, which leads to incorrect answers to these questions. Correct specification is accomplished here using enumerative techniques. The answers to the questions are updated and recommended strategies are revised and expanded. Results and findings are presented along with those from Tan's article for comparison.

TABLE 1: An example of a battle

Roll #	No. of armies		No. of dice rolled		Outcome of the dice		No. of losses	
	attacker	defender	attacker	defender	attacker	defender	attacker	defender
1	4	3	3	2	5, 4, 3	6, 3	1	1
2	3	2	3	2	5, 5, 3	5, 5	2	0
3	1	2	1	2	6	4, 3	0	1
4	1	1	1	1	5	6	1	0
5	0	1						

The Markov chain The object for a player in *RISK* is to conquer the world by occupying all 42 countries, thereby destroying all armies of the opponents. The rules of *RISK* are straightforward and many readers may need no review. Newcomers are referred to Tan's article where a clear and concise presentation can be found. Tan's Table 1 is reproduced here for convenience. It shows the progress of a typical battle over a country, with the defender prevailing after five rolls. This table also serves as a reminder of the number of dice rolled in various situations—never more than three for the attacker, and never more than two for the defender.

Following Tan's notation, let A denote the number of attacking armies and D the number of defending armies at the beginning of a battle. The state of the battle at any time can be characterized by the number of attacking and defending armies remaining. Let $X_n = (a_n, d_n)$ be the state of the battle after the nth roll of the dice, where a_n and d_n denote the number of attacking and defending armies remaining. The initial state is $X_0 = (A, D)$. The probability that the system goes from one state at turn n to another state at turn $n+1$, given the history before turn n, depends only on (a_n, d_n), so that $\{X_n : n = 0, 1, 2, \ldots\}$ forms a Markov chain:

$$\Pr\left[X_{n+1} = (a_{n+1}, d_{n+1}) \mid x_n, x_{n-1}, \ldots, x_1, x_0\right] = \Pr\left[X_{n+1} = (a_{n+1}, d_{n+1}) \mid x_n\right]$$

The AD states where both a and d are positive are transient. The $D + A$ states where either $a = 0$ or $d = 0$ are absorbing. Let the $AD + (D + A)$ possible states be ordered so that the AD transient states are followed by the $D + A$ absorbing states. Let the transient states be ordered

$$\left\{(1, 1), (1, 2), \ldots, (1, D), (2, 1), (2, 2), \ldots, (2, D), \ldots, (A, D)\right\}$$

and the absorbing states

$$\left\{(0, 1), (0, 2), \ldots, (0, D), (1, 0), (2, 0), \ldots, (A, 0)\right\}.$$

Under this ordering, the transition probability matrix takes the simple form

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix},$$

where the $(AD) \times (AD)$ matrix Q contains the probabilities of going from one transient state to another, and the $(AD) \times (D + A)$ matrix R contains the probabilities of going from a transient state into an absorbing state.

The transition probability matrix, *P* It turns out that P contains only 14 distinct probabilities, having to do with how many dice are rolled and how many armies lost as a result. Let π_{ijk} denote the probability that the defender loses k armies when rolling j dice against an attacker rolling i dice, as given in Table 2. To obtain the π_{ijk}, the joint probabilities associated with the best and second best roll from 2 or 3 six-sided dice need to be quantified. Let Y_1, Y_2, Y_3 denote the unordered outcome for an attacker when rolling three dice and let W_1, W_2 denote the unordered outcome for an attacker when rolling two dice. Let Z_1, Z_2 denote the unordered outcome for a defender rolling two dice. Then Y_1, Y_2, Y_3 and W_1, W_2 and Z_1, Z_2 are random samples from the discrete uniform distribution on the integers 1 through 6:

$$\Pr(Y_j = y) = \begin{cases} \frac{1}{6} & \text{for } y = 1, 2, 3, 4, 5, 6 \\ 0 & \text{else.} \end{cases}$$

When order is taken into account and denoted using superscripts, as in $Y^{(1)} \geq Y^{(2)} \geq Y^{(3)}$, the ordered random variables are called *order statistics.* The joint distributions of order statistics needed for specification of π_{ijk} can be obtained using straightforward techniques of enumeration. When two dice are rolled, the joint distribution of $(Z^{(1)}, Z^{(2)})$ is

$$\Pr(Z^{(1)} = z^{(1)}, Z^{(2)} = z^{(2)}) = \begin{cases} \frac{1}{36} & \text{for } z^{(1)} = z^{(2)} \\ \frac{2}{36} & \text{for } z^{(1)} > z^{(2)} \\ 0 & \text{else,} \end{cases}$$

and the *marginal* distribution of the best roll $Z^{(1)}$ is obtained by summing the joint distribution over values of $z^{(2)}$:

$$\Pr(Z^{(1)} = z^{(1)}) = \begin{cases} \dfrac{2z^{(1)} - 1}{36} & \text{for } z^{(1)} = 1, 2, 3, 4, 5, 6. \end{cases}$$

When three dice are rolled, the pertinent distribution of the best two rolls is

$$\Pr(Y^{(1)} = y^{(1)}, Y^{(2)} = y^{(2)}) = \begin{cases} \frac{3y^{(1)}-2}{216} & \text{for } y^{(1)} = y^{(2)} \\ \frac{6y^{(2)}-3}{216} & \text{for } y^{(1)} > y^{(2)} \\ 0 & \text{else,} \end{cases}$$

and the marginal distribution of the best roll is

$$\Pr(Y^{(1)} = y^{(1)}) = \begin{cases} \dfrac{1 - 3y^{(1)} + 3(y^{(1)})^2}{216} & \text{for } y^{(1)} = 1, 2, 3, 4, 5, 6. \end{cases}$$

All of the probabilities are 0 for arguments that are not positive integers less than or equal to 6. The joint distribution of $W^{(1)}$ and $W^{(2)}$ is the same as that for $Z^{(1)}$ and $Z^{(2)}$.

The marginal distributions given in Tan's article can be obtained directly from these joint distributions. However, the marginal distributions alone are not sufficient to correctly specify the probabilities of all 14 possible outcomes. In obtaining probabilities such as π_{322} and π_{320}, Tan's mistake is in assuming the independence of events such as $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$. Consider π_{322}. Tan's calculation proceeds below:

$$\begin{aligned}
\pi_{322} &= \Pr(Y^{(1)} > Z^{(1)} \cap Y^{(2)} > Z^{(2)}) \\
&= \Pr(Y^{(1)} > Z^{(1)}) \Pr(Y^{(2)} > Z^{(2)}) \\
&= (0.471)(0.551) \\
&= 0.259.
\end{aligned}$$

The correct probability can be written in terms of the joint distributions for ordered outcomes from one, two, or three dice. For example,

$$\begin{aligned}
\pi_{322} &= \Pr(Y^{(1)} > Z^{(1)}, Y^{(2)} > Z^{(2)}) \\
&= \sum_{z_1=1}^{5} \sum_{z_2=1}^{z_1} \Pr(Y^{(1)} > z_1, Y^{(2)} > z_2) \Pr(Z^{(1)} = z_1, Z^{(2)} = z_2) \\
&= \sum_{z_1=1}^{5} \sum_{z_2=1}^{z_1} \sum_{y_1=z_1+1}^{6} \sum_{y_2=z_2+1}^{y_1} \Pr(Y^{(1)} = y_1, Y^{(2)} = y_2) \Pr(Z^{(1)} = z_1, Z^{(2)} = z_2) \\
&= \frac{2890}{7776} \\
&= 0.372.
\end{aligned}$$

Note that those events in this quadruple sum for which an argument with a subscript of 2 exceeds an argument with the same letter and subscript 1 have probability zero.

The probabilities π_{ijk} that make up the transition probability matrix P can be obtained similarly using the joint distributions for $Y^{(1)}, Y^{(2)}$, for $Z^{(1)}, Z^{(2)}$, and for $W^{(1)}, W^{(2)}$. The probabilities themselves, rounded to the nearest 0.001, are listed in Table 2.

TABLE 2: Probabilities making up the transition probability matrix

i	j	Event	Symbol	Probability	Tan's value
1	1	Defender loses 1	π_{111}	$15/36 = 0.417$	0.417
1	1	Attacker loses 1	π_{110}	$21/36 = 0.583$	0.583
1	2	Defender loses 1	π_{121}	$55/216 = 0.255$	0.254
1	2	Attacker loses 1	π_{120}	$161/216 = 0.745$	0.746
2	1	Defender loses 1	π_{211}	$125/216 = 0.579$	0.578
2	1	Attacker loses 1	π_{210}	$91/216 = 0.421$	0.422
2	2	Defender loses 2	π_{222}	$295/1296 = 0.228$	0.152
2	2	Each lose 1	π_{221}	$420/1296 = 0.324$	0.475
2	2	Attacker loses 2	π_{220}	$581/1296 = 0.448$	0.373
3	1	Defender loses 1	π_{311}	$855/1296 = 0.660$	0.659
3	1	Attacker loses 1	π_{310}	$441/1296 = 0.340$	0.341
3	2	Defender loses 2	π_{322}	$2890/7776 = 0.372$	0.259
3	2	Each lose 1	π_{321}	$2611/7776 = 0.336$	0.504
3	2	Attacker loses 2	π_{320}	$2275/7776 = 0.293$	0.237

The probability of winning a battle Given any initial state, the system will, with probability one, eventually make a transition to an absorbing state. For a transient state i, call $f_{ij}^{(n)}$ the probability that the first (and final) visit to absorbing state j occurs on the nth turn:

$$f_{ij}^{(n)} = \Pr(X_n = j, X_k \neq j \text{ for } k = 1, \ldots, n-1 \mid X_0 = i).$$

Let the $AD \times (D + A)$ matrix of these *first transition* probabilities be denoted by $F^{(n)}$. In order for the chain to begin at state i and enter state j at the nth turn, the first $n - 1$ transitions must be among the transient states and the nth must be from a transient state to an absorbing state so that $F^{(n)} = Q^{n-1}R$. The system proceeds for as many turns as are necessary to reach an absorbing state. The probability that the system goes from transient state i to absorbing state j is just the sum $f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$. The $AD \times (D + A)$ matrix of probabilities for all of these $D + A$ absorbing states can be obtained from

$$F = \sum_{n=1}^{\infty} F^{(n)} = \sum_{n=1}^{\infty} Q^{n-1}R = (I - Q)^{-1}R.$$

If the system ends in one of the last A absorbing states then the attacker wins; if it ends in one of the first D absorbing states, the defender wins. Since the initial state of a battle is the $i = AD$th state using the order established previously, the probability that the attacker wins is just the sum of the entries in the last (or ADth) row of the submatrix of the last A columns of F:

$$\Pr\big(\text{Attacker wins} \mid X_0 = (A, D)\big) = \sum_{j=D+1}^{D+A} f_{AD,j}$$

and

$$\Pr\big(\text{Defender wins} \mid X_0 = (A, D)\big) = \sum_{j=1}^{D} f_{AD,j}.$$

The row sums of F are unity, which confirms that the system always ends in one of the $D + A$ absorbing states.

TABLE 3: Probability that the attacker wins

D \ A	1	2	3	4	5	6	7	8	9	10
1	0.417	0.106	0.027	0.007	0.002	0.000	0.000	0.000	0.000	0.000
2	0.754	0.363	0.206	0.091	0.049	0.021	0.011	0.005	0.003	0.001
3	0.916	0.656	0.470	0.315	0.206	0.134	0.084	0.054	0.033	0.021
4	0.972	0.785	0.642	0.477	0.359	0.253	0.181	0.123	0.086	0.057
5	0.990	0.890	0.769	0.638	0.506	0.397	0.297	0.224	0.162	0.118
6	0.997	0.934	0.857	0.745	0.638	0.521	0.423	0.329	0.258	0.193
7	0.999	0.967	0.910	0.834	0.736	0.640	0.536	0.446	0.357	0.287
8	1.000	0.980	0.947	0.888	0.818	0.730	0.643	0.547	0.464	0.380
9	1.000	0.990	0.967	0.930	0.873	0.808	0.726	0.646	0.558	0.480
10	1.000	0.994	0.981	0.954	0.916	0.861	0.800	0.724	0.650	0.568

The matrix F is used to obtain Table 3, a matrix of victory probabilities for a battle between an attacker with i armies and a defender with j armies for values of i and j

not greater than 10. Some of these are shown graphically in FIGURE 1, along with some for higher numbers of attacking armies.

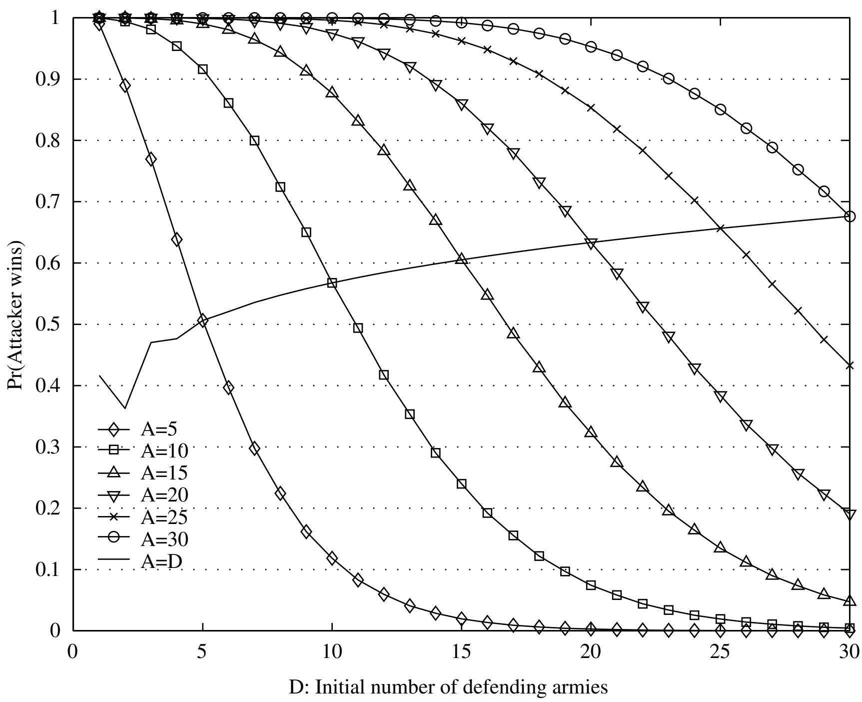

Figure 1 Attacker's winning probabilities at various strengths

Expected losses The expected values and variances for the losses that the attacker and defender will suffer in a given battle can also be found from F. Let L_A and L_D denote the respective losses an attacker and defender will suffer during a given battle given the initial state $X_0 = (A, D)$. Let $R_D = D - L_D$ and $R_A = A - L_R$ denote the number of armies remaining for the attacker and defender respectively. The probability distributions for R_D and R_A can be obtained from the last row of F:

$$Pr(R_D = k) = \begin{cases} f_{AD,k} & \text{for } k = 1, \ldots, D \\ 0 & \text{else} \end{cases}$$

and

$$\Pr(R_A = k) = \begin{cases} f_{AD,D+k} & \text{for } k = 1, \ldots, A \\ 0 & \text{else.} \end{cases}$$

For example, suppose $A = D = 5$. In this case, the 25th row of the 25×10 matrix F gives the probabilities for the $D + A = 10$ absorbing states:

$$F_{25,\cdot} = (0.068, 0.134, 0.124, 0.104, 0.064, 0.049, 0.096, 0.147, 0.124, 0.091).$$

The mean and standard deviation for the defender's loss in the $A = D = 5$ case are $E(L_D) = 3.56$ and $SD(L_D) = 1.70$. For the attacker, they are $E(L_A) = 3.37$ and $SD(L_A) = 1.83$. Plots of expected losses for values of A and D between 5 and 20 are given in FIGURE 2. This plot shows that the attacker has an advantage in the sense that expected losses are lower than for the defender, provided the initial number of attacking armies is not too small.

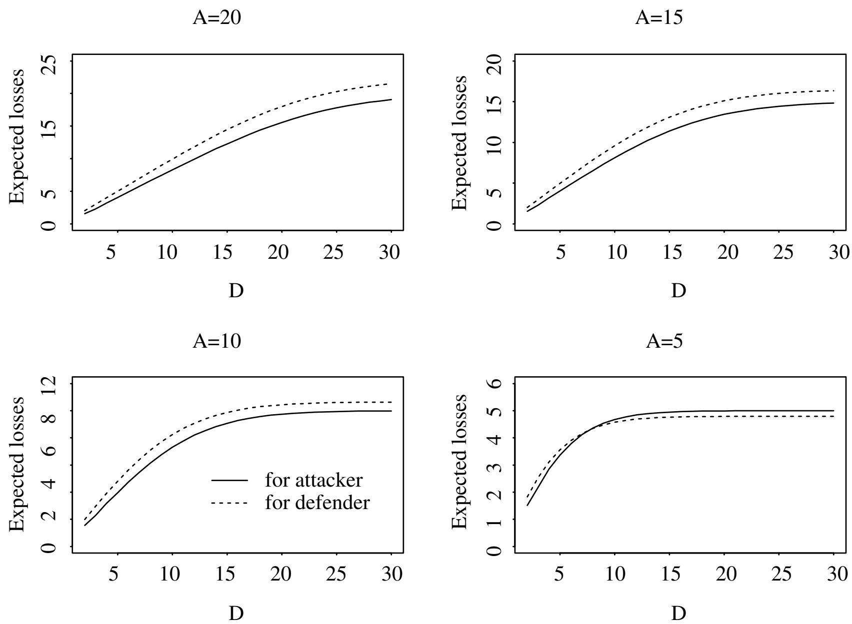

Figure 2 Expected losses for attacker and for defender

Conclusion and recommendations The chances of winning a battle are considerably more favorable for the attacker than was originally suspected. The logical recommendation is then for the attacker to be more aggressive. Inspection of FIGURE 1 shows that when the number of attacking and defending armies is equal ($A = D$), the probability that the attacker ends up winning the territory exceeds 50%, provided the initial stakes are high enough (at least 5 armies each, initially.) This is contrary to Tan's assertion that that this probability is less than 50% because "in the case of a draw, the defender wins" in a given roll of the dice. When $A = D$, FIGURE 2 indicates that the attacker also suffers fewer losses on average than the defender, provided A is not small. With the innovation of several new versions of *RISK* further probabilistic challenges have arisen. *RISK II* enables players to attack simultaneously rather than having to wait for their turn and involves single rolls of nonuniformly distributed dice. The distribution of the die rolled by an attacker or defender depends on the number of armies the player has stationed in an embattled country. The Markovian property of a given battle still holds, but the entries of the transition probability matrix P are different. Further, decisions about whether or not to attack should be made with the knowledge that attacks cannot be called off as in the original *RISK*.

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