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ASYMPTOTIC EXPANSIONS OF POSTERIOR DISTRIBUTIONS IN NONREGULAR CASES

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Abstract. We study the asymptotic behaviour of the posterior distributions for a one-parameter family of discontinuous densities. It is shown that a suitably centered and normalized posterior converges almost surely to an exponential limit in the total variation norm. Further, asymptotic expansions for the density, distribution function, moments and quantiles of the posterior are also obtained. It is to be noted that, in view of the results of Ghosh *et al.* (1994, *Statistical Decision Theory and Related Topics V*, 183–199, Springer, New York) and Ghosal *et al.* (1995, *Ann. Statist.*, **23**, 2145–2152), the nonregular cases considered here are essentially the only ones for which the posterior distributions converge. The results obtained here are also supported by a simulation experiment.

Key words and phrases: Asymptotic expansion, posterior distributions, nonregular cases.

1. Introduction

It is well known that in the usual "regular" cases, for a wide variety of priors, the posterior, suitably normalized and centered at the maximum likelihood estimate, tends to the standard normal distribution. Such a limiting behaviour of the posterior was studied by several authors including Le Cam (1953), Bickel and Yahav (1969), Walker (1969) and Johnson (1970). Johnson (1970) obtained asymptotic expansions for posterior distributions in regular cases with a normal distribution as the leading term.

Recently Ghosh *et al.* (1994) considered a general situation including the regular and a wide variety of nonregular cases. They considered the general set up of Ibragimov and Has'minskii (1981) and obtained a necessary condition for the existence of a limit (in probability) of suitably centered posterior in terms

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of the limiting likelihood ratio process. In Ghosal *et al.* (1995) this condition is also shown to be sufficient for the existence of an in-probability limit. Ghosh *et al.* (1994) applied their results on different classes of nonregular examples and it turned out that for many of the nonregular examples, a posterior limit does not exist.

In this paper, we consider the nonregular cases which, in view of the results of Ghosh *et al.* (1994) and Ghosal *et al.* (1995), are essentially the only ones for which posterior limits exist. By results obtained in Ghosal *et al.* (1995), an in-probability limit of the posterior exists for this class. In Section 2 of this paper, we obtain an almost sure limit of the posterior distribution. We further extend this result in Section 3 where almost sure asymptotic expansions of the posterior density and the posterior distribution function are obtained. Asymptotic expansions for the posterior moments and quantiles are presented in Section 4. In Section 5, we assess the quality of the approximations by means of a simulation experiment.

2. Almost sure limit of posterior distributions

Let $X_1, X_2, \ldots, X_n$ be independent observations with a common distribution P_θ possessing a density $f(x,\theta)$ on $\mathbb{R}$ with respect to the Lebesgue measure where $\theta \in \Theta$, an open interval (bounded or unbounded) in $\mathbb{R}$. We fix $\theta_0 \in \Theta$ which may be regarded as the true parameter point. We assume that for all $\theta \in \Theta$, $f(\cdot,\theta)$ is strictly positive in a closed interval (bounded or unbounded) $S(\theta) := [a_1(\theta), a_2(\theta)]$ depending on θ and is zero outside $S(\theta)$. It is permitted that one of the endpoints is free of θ and may be plus or minus infinity (see the examples of this section). In view of the results of Ghosh *et al.* ((1994), Theorem 2.4 and Example 3.2), in order to have a limit of the posterior, it is necessary that the sets $S(\theta)$ are either increasing or decreasing in θ. The case where $S(\theta)$ increases with θ may be reduced to the case where $S(\theta)$ decreases by the reparametrization $\theta \mapsto (-\theta)$. We therefore consider only the latter, namely the case where $a_1(\theta)$ is increasing and $a_2(\theta)$ is decreasing in θ. Moreover, we assume that these functions are strictly monotone unless they are infinite or free from θ.

We now make the following assumptions on the density $f(x,\theta)$.

(A1) The endpoints a_1 and a_2 are continuously differentiable functions of θ if these are not minus or plus infinity (as mentioned above, one of the endpoints may be infinity).

(A2) On the set $\{(x,\theta) : x \in S(\theta)\}$, $f(x,\theta)$ is jointly continuous in (x,θ).

(A3) For each x, $\log f(x,\theta)$ is twice differentiable in θ on $\{a_1(\theta) < x < a_2(\theta)\}$. Further, we have the following:

(a) For all $\theta \in \Theta$, $c(\theta) := E_\theta[(\partial/\partial\theta)\log f(X_1,\theta)]$ is finite.

(b) There exist a neighbourhood N_{θ_0} of θ_0 and a P_{θ_0}-integrable function $H_{\theta_0}(x)$ such that for all $\theta \in N_{\theta_0}$ and $x \in (a_1(\theta), a_2(\theta))$, $|(\partial^2/\partial\theta^2)\log f(x,\theta)| < H_{\theta_0}(x)$.

(A4) For all sufficiently large $\lambda > 0$,

$$E_{\theta_0}[\sup\{\log(f(X_1,\theta)/f(X_1,\theta_0)) : \theta < \theta_0 - \lambda, \theta \in \Theta\}] < 0.$$

(A5) As $\rho \to 0$, $E_{\theta_0}\log f(X_1,\theta,\rho) \to E_{\theta_0}\log f(X_1,\theta)$, where $f(x,\theta,\rho) = \sup\{f(x,\theta') : |\theta - \theta'| \le \rho\}$.

Several important examples fall in the above set up.

1. Location family: $f(x,\theta) = f(x-\theta)$, $\theta \in \mathbb{R}$, where $f(\cdot)$ is a density supported on $[0,\infty)$ with $f(0) > 0$. Here $a_1(\theta) = \theta$ and $a_2(\theta) \equiv \infty$. A particular example is $f(x) = e^{-x}$, $x > 0$.

2. Uniform distributions supported on (i) $[0,\theta]$, $\theta > 0$; (ii) $[-\theta,\theta]$, $\theta > 0$; (iii) $[\theta, 1/\theta]$, $0 < \theta < 1$.

It may be noted that for distributions like $U[\theta,\theta+1]$ or $U[\theta,2\theta]$, the supports are not monotone and hence these do not fall in our set up.

3. Truncation family: $f(x,\theta) = g(x)/\bar{G}(\theta)$, $x > \theta$, where $g(\cdot)$ is a density supported on $[0,\infty)$ which is positive on $(0,\infty)$ and $\bar{G}(x) = \int_x^\infty g(t)dt$.

It can be easily seen that the set $\{a_1(\theta) \le X_i \le a_2(\theta), i = 1,2,\ldots,n\}$ can be expressed as $\{\hat{\theta}_n(X_1,\ldots,X_n) \ge \theta\}$ where $\hat{\theta}_n = \min\{a_1^{-1}(X_{\min}), a_2^{-1}(X_{\max})\}$ (if $a_1(\theta)$ is free of θ or $-\infty$, then we interpret the above minimum as $a_2^{-1}(X_{\max})$ while it is interpreted as $a_1^{-1}(X_{\min})$ if $a_2(\theta)$ is free of θ or ∞). It is to be noted that $\hat{\theta}_n$ is defined a.s. $[P_{\theta_0}]$ for all sufficiently large n and $\hat{\theta}_n \to \theta_0$ a.s. $[P_{\theta_0}]$ as $n \to \infty$. Further, using a dominated convergence argument and Leibniz's rule of differentiation of an integral, it is easy to see that $c(\theta_0) = a_1'(\theta_0)f(a_1(\theta_0),\theta_0) - a_2'(\theta_0)f(a_2(\theta_0),\theta_0) > 0$, where prime stands for derivative. Here, we use the convention that $a_i'(\theta) = 0$ if $a_i(\cdot)$ is infinite, $i = 1,2$. The phrase "for all sufficiently large n" will be often omitted.

We now consider a prior probability distribution on Θ with density $\pi(\cdot)$ with respect to the Lebesgue measure. The posterior density of θ given the observations $X_1, X_2, \ldots, X_n$ is given by

$$\pi_n(\theta) = \pi_n(\theta \mid X_1,\ldots,X_n) = \frac{\prod_{i=1}^n f(X_i,\theta)\pi(\theta)}{\int_{\mathbb{R}} \prod_{i=1}^n f(X_i,\eta)\pi(\eta)d\eta}. \tag{2.1}$$

Set

$$Z_n(u) = \prod_{i=1}^n \frac{f(X_i,\hat{\theta}_n + u/\sigma_n)}{f(X_i,\hat{\theta}_n)}, \qquad \sigma_n = \sum_{i=1}^n (\partial/\partial\theta)\log f(X_i,\hat{\theta}_n).$$

The posterior density of $u = \sigma_n(\theta - \hat{\theta}_n)$ given $X_1,\ldots,X_n$ is given by

$$\pi_n^*(u) = \pi_n^*(u \mid X_1,\ldots,X_n) = \frac{Z_n(u)\pi(\hat{\theta}_n + u/\sigma_n)}{\int_{\mathbb{R}} Z_n(v)\pi(\hat{\theta}_n + v/\sigma_n)dv}. \tag{2.2}$$

It will be shown in the proof of Lemma 2.1 that $\sigma_n/n \to c(\theta_0)$ a.s. Thus by definition of $\hat{\theta}_n$, it is immediate that $\pi_n^*(u) = 0$ for $u > 0$.

In this section, we find an almost sure limit of the normalized and centered posterior $\pi_n^*(\cdot)$. The following theorem is the main result of this section and states that the posterior converges to an exponential distribution.

THEOREM 2.1. *Under Assumptions* (A1)–(A5), *for any prior probability density* $\pi(\cdot)$ *over* Θ *which is positive and continuous at* θ_0, *we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}} |\pi_n^*(u) - \pi^*(u)|du = 0 \qquad a.s., \tag{2.3}$$

where $\pi_n^*(u) = \pi_n^*(u \mid X_1, \ldots, X_n)$ *is the posterior density of* $u = \sigma_n(\theta - \hat{\theta}_n)$ *given the observations* $X_1, \ldots, X_n$ *and* $\pi^*(u) = e^u I\{u < 0\}$.

We first prove the following lemmas.

LEMMA 2.1. *There exists* $\delta > 0$ *such that with probability one,*

$$\log Z_n(u) \le u/2, \qquad -n\delta \le u < 0,$$

for all sufficiently large n.

LEMMA 2.2. *For any* $\delta > 0$, *there exists* $\varepsilon > 0$ *such that with probability one,*

$$\log Z_n(u) < -n\varepsilon, \qquad u < -n\delta$$

for all sufficiently large n.

PROOF OF LEMMA 2.1. Expanding by Taylor's theorem we have

$$\sigma_n/n = n^{-1}\sum_{i=1}^{n}(\partial/\partial\theta)\log f(X_i,\theta_0) + n^{-1}(\hat{\theta}_n - \theta_0)\sum_{i=1}^{n}(\partial^2/\partial\theta^2)\log f(X_i,\theta_n')$$

where θ_n' lies between θ_0 and $\hat{\theta}_n$. Since $\hat{\theta}_n \to \theta_0$ a.s., by (A3) and the Strong Law of Large Numbers (SLLN), $\sigma_n/n \to c(\theta_0)$ a.s.

We now use the Taylor expansion once again to get

$$\log Z_n(u) = \frac{u}{\sigma_n}\sum_{i=1}^{n}(\partial/\partial\theta)\log f(X_i,\hat{\theta}_n) + \frac{u^2}{2\sigma_n^2}\sum_{i=1}^{n}(\partial^2/\partial\theta^2)\log f(X_i,\theta_n^*) \tag{2.4}$$

where θ_n^* lies between $\hat{\theta}_n$ and $\hat{\theta}_n + u/\sigma_n$. Since $\sigma_n/n \to c(\theta_0)$ a.s., we can choose $\delta > 0$ such that for all u in $\{|u| \le n\delta\}$, $\theta_n^* \in N_{\theta_0}$. Now the first term on the right hand side (RHS) of (2.4) is equal to u. The second term in absolute value is dominated by $(|u|n\delta/\sigma_n^2)\sum_{i=1}^{n} H(X_i)$ for all u satisfying $|u| \le n\delta$. Thus we can choose δ so small that the second term is dominated by $|u|/2$ for all $-n\delta \le u < 0$. The result now follows. □

PROOF OF LEMMA 2.2. The idea of the proof given below is essentially due to Wald (1949). We write

$$\begin{aligned} n^{-1}\log Z_n(u) &= n^{-1}\sum_{i=1}^{n}\log\frac{f(X_i,\hat{\theta}_n + u/\sigma_n)}{f(X_i,\theta_0)} - n^{-1}\sum_{i=1}^{n}\log\frac{f(X_i,\hat{\theta}_n)}{f(X_i,\theta_0)} \\ &= A_n + B_n \qquad \text{(say)}. \end{aligned} \tag{2.5}$$

It is easy to prove that $B_n \to 0$ a.s. Also, for $u < -n\delta$, we have $\hat{\theta}_n + \sigma_n^{-1}u - \theta_0 < -\delta/(2c(\theta_0))$ and therefore,

$$A_n \le \sup_{\theta \le \theta_0 - \delta/(2c(\theta_0))} n^{-1}\sum_{i=1}^{n}\log\frac{f(X_i,\theta)}{f(X_i,\theta_0)}. \tag{2.6}$$

By Assumption (A4) we can get $\lambda_0 > \delta/(2c(\theta_0))$ such that

$$E_{\theta_0}\left[\sup_{\theta<\theta_0-\lambda_0} \log \frac{f(X_1,\theta)}{f(X_1,\theta_0)}\right] < 0. \tag{2.7}$$

Set $\Theta_0 = \{\theta \in \Theta : \theta < \theta_0 - \lambda_0\}$ and $\Theta_1 = \{\theta \in \Theta : \theta_0 - \lambda_0 \le \theta \le \theta_0 - \delta/(2c(\theta_0))\}$.

For each $\theta \in \Theta_1$, we can get $\rho_\theta > 0$ such that $E_{\theta_0} \log f(X_1,\theta,\rho_\theta) < E_{\theta_0} \log f(X_1,\theta_0)$ where $f(x,\theta,\rho_\theta)$ is as defined earlier (see (A5)). This is possible by (A5) and the fact that $E_{\theta_0} \log f(X_1,\theta) < E_{\theta_0} \log f(X_1,\theta_0)$ for $\theta \ne \theta_0$. Since the set Θ_1 is compact, there exist a finite number of points $\theta_1, \theta_2, \ldots, \theta_k \in \Theta_1$ such that $\bigcup_{j=1}^k (\theta_j - \rho_{\theta_j}, \theta_j + \rho_{\theta_j})$ covers Θ_1 and

$$E_{\theta_0}[\log f(X_1,\theta_j,\rho_{\theta_j})] < E_{\theta_0}[\log f(X_1,\theta_0)], \qquad j = 1,2,\ldots,k. \tag{2.8}$$

Now for all $u < -n\delta$, we have from (2.6)

$$\begin{aligned} A_n &\le \sup\left\{ n^{-1}\sum_{i=1}^n \log\frac{f(X_i,\theta)}{f(X_i,\theta_0)} : \theta \in \Theta_0 \bigcup \left[\bigcup_{j=1}^k (\theta_j - \rho_{\theta_j}, \theta_j + \rho_{\theta_j})\right]\right\} \\ &\le \max\left\{ n^{-1}\sum_{i=1}^n \sup_{\theta\in\Theta_0}\left[\log\frac{f(X_i,\theta)}{f(X_i,\theta_0)}\right],\right. \\ &\qquad\qquad \left. n^{-1}\max\left\{\sum_{i=1}^n \log\frac{f(X_i,\theta_j,\rho_{\theta_j})}{f(X_i,\theta_0)}, j = 1,2,\ldots,k\right\}\right\}. \end{aligned}$$

Using the inequalities (2.7) and (2.8) and the SLLN, we can get $\varepsilon > 0$ such that with probability one $A_n < -\varepsilon$. This proves the result. □

Proof of Theorem 2.1. From (2.4), by Assumption (A3) and an application of the SLLN, $Z_n(u) \to e^u$ a.s. for each fixed $u < 0$, while $Z_n(u) = 0$ for $u > 0$. We write

$$\begin{aligned} &\int_{\mathbb{R}} |\pi(\hat\theta_n + u/\sigma_n) Z_n(u) - \pi(\theta_0)\pi^*(u)| du \\ &\quad = \int_{-n\delta}^0 |\pi(\hat\theta_n + u/\sigma_n) Z_n(u) - \pi(\theta_0) e^u| du \\ &\qquad + \int_{-\infty}^{-n\delta} |\pi(\hat\theta_n + u/\sigma_n) Z_n(u) - \pi(\theta_0) e^u| du \\ &\quad = I_{1n} + I_{2n} \qquad \text{(say)}, \end{aligned} \tag{2.9}$$

where $\delta > 0$ is as in Lemma 2.1. Now

$$I_{1n} \le \int_{-n\delta}^0 Z_n(u)|\pi(\hat\theta_n + u/\sigma_n) - \pi(\theta_0)| du + \int_{-n\delta}^0 \pi(\theta_0)|Z_n(u) - e^u| du.$$

By Lemma 2.1 and the dominated convergence theorem both the above integrals converge to zero a.s. (for sufficiently small δ). Also we have

$$I_{2n} \leq \int_{-\infty}^{-n\delta} Z_n(u)\pi(\hat{\theta}_n + u/\sigma_n)du + \int_{-\infty}^{-n\delta} \pi(\theta_0)e^u du. \tag{2.10}$$

The second term on the RHS of (2.10) obviously goes to zero, while the first term goes to zero a.s. by Lemma 2.2. Thus we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} |\pi(\hat{\theta}_n + u/\sigma_n)Z_n(u) - \pi(\theta_0)\pi^*(u)|du = 0 \quad \text{a.s.} \tag{2.11}$$

Therefore,

$$D_n := \int_{\mathbb{R}} \pi(\hat{\theta}_n + u/\sigma_n)Z_n(u)du \to \int_{\mathbb{R}} \pi(\theta_0)\pi^*(u)du = \pi(\theta_0) \quad \text{a.s.} \tag{2.12}$$

and hence

$$\begin{aligned}\int_{\mathbb{R}} |\pi_n^*(u) - \pi^*(u)|du \leq & \int_{\mathbb{R}} D_n^{-1}|Z_n(u)\pi(\hat{\theta}_n + u/\sigma_n) - \pi(\theta_0)\pi^*(u)|du \\ & + \int_{\mathbb{R}} |D_n^{-1}\pi(\theta_0) - 1|\pi^*(u)du\end{aligned}$$

and these two terms converge to zero a.s. by (2.11) and (2.12). □

Before we end this section, we indicate an alternative way of proving (2.3). Consider the posterior probability $P_{n,\delta} = \int_{\hat{\theta}_n-\delta}^{\hat{\theta}_n+\delta} \pi_n(\theta)d\theta$, $\delta > 0$. As noted earlier, $\pi_n(\theta) = 0$ for $\theta > \hat{\theta}_n$ where $\pi_n(\theta)$ is the posterior density as given in (2.1). For $\hat{\theta}_n - \delta \leq \theta \leq \hat{\theta}_n$, we write

$$\begin{aligned}\pi_n(\theta) &= \pi_n(\hat{\theta}_n)\exp\{\log\pi_n(\theta) - \log\pi_n(\hat{\theta}_n)\} \\ &= \pi_n(\hat{\theta}_n)\exp\{(\theta - \hat{\theta}_n)\sum_{i=1}^{n}(\partial/\partial\theta)\log f(X_i, \theta_n') + \log\pi(\theta) - \log\pi(\hat{\theta}_n)\},\end{aligned}$$

where θ_n' lies between θ and $\hat{\theta}_n$. Then proceeding as in Chen (1985), we can prove the following.

LEMMA 2.3. *Assume* (A1)–(A3). *Then* $P_{n,\delta} \to 1$ *a.s. for all* $\delta > 0$ *if and only if* $\pi_n(\hat{\theta}_n)\sigma_n^{-1} \to 1$ *a.s.*

It is well known that under a very general set up, the posterior is consistent, i.e., the posterior probability of the set $(\theta_0 - \delta, \theta_0 + \delta)$ goes to one a.s. for all $\delta > 0$. Ghosal *et al.* (1995) presented a proof of this result under certain conditions which were referred to as Conditions (IH) (after Ibragimov and Has'minskii (1981)). Ibragimov and Has'minskii ((1981), Chapter V) considered densities with jumps and gave sufficient conditions for Conditions (IH) to hold. Thus, under those conditions, we have posterior consistency and since $\hat{\theta}_n \to \theta_0$ a.s., this implies $P_{n,\delta} \to 1$ a.s. for all $\delta > 0$. Therefore, by Lemma 2.3 we have $\pi_n(\hat{\theta}_n)\sigma_n^{-1} \to 1$ a.s. Now

$$\begin{aligned}\log \frac{\pi_n(\hat{\theta}_n + u/\sigma_n)}{\pi_n(\hat{\theta}_n)} &= \sum_{i=1}^{n} \log \frac{f(X_i, \hat{\theta}_n + u/\sigma_n)}{f(X_i, \hat{\theta}_n)} + \log \pi(\hat{\theta}_n + u/\sigma_n) - \log \pi(\hat{\theta}_n) \\ &- u\sigma_n^{-1} \sum_{i=1}^{n} (\partial/\partial\theta) \log f(X_i, \theta_n'') \\ &+ \log \pi(\hat{\theta}_n + u/\sigma_n) - \log \pi(\hat{\theta}_n)\end{aligned}$$

where θ_n'' lies between $\hat{\theta}_n + u/\sigma_n$ and $\hat{\theta}_n$. Thus for any $u < 0$, $\log(\pi_n(\hat{\theta}_n + u/\sigma_n)/\pi_n(\hat{\theta}_n)) \to u$; consequently with probability one $\pi_n^*(u) \to e^u$ for each fixed $u < 0$. An application of Scheffe's Theorem now proves the result. □

3. Asymptotic expansions of posterior distributions

We consider the set up and assumptions of Section 2. In addition to (A1) (A5), we also make the following assumptions.

(A6) For each x, $\log f(x, \theta)$ is $(r+2)$ times continuously differentiable in θ on $\{a_1(\theta) < x < a_2(\theta)\}$. Further, there is a neighbourhood N_{θ_0} of θ_0 and P_{θ_0}-integrable functions $H_k(x)$, $k = 2, \dots, r+2$, such that

$$|(\partial^k/\partial\theta^k) \log f(x, \theta)| \le H_k(x), \qquad k = 2, \dots, r+2,$$

for all $\theta \in N_{\theta_0}$ and $x \in (a_1(\theta), a_2(\theta))$.

(A7) The prior density $\pi(\theta)$ is $(r+1)$ times continuously differentiable in a neighbourhood of θ_0 and $\pi(\theta_0) > 0$.

The following theorem gives an asymptotic expansion of the posterior distribution.

THEOREM 3.1. *Under Assumptions* (A1)–(A7), *we have*

$$\int_{-\infty}^{0} |\pi_n^*(u) - e^u C_n^{-1} \sum_{k=0}^{r} \alpha_k(u) n^{-k}| du = O(n^{-(r+1)}) \qquad a.s.$$

where $C_n = \sum_{l+m \le r} (-1)^l c_{lm} (l + 2m)! n^{-(l+m)}$, $\alpha_k(u) = \alpha_k(u, \boldsymbol{x}) = \sum_{m=0}^{k} c_{k-m,m} u^{k+m}$, $c_{lm} = c_{lm}(\boldsymbol{x})$, $l, m = 0, 1, \dots$, *are constants as defined in* (3.10) *below and* $\boldsymbol{x}$ *denotes the sample sequence.*

Remark 3.1. Considering the quotient series $C_n^{-1} \sum_{k=0}^{r} \alpha_k(u) n^{-k}$, the theorem can be restated as

$$\int_{-\infty}^{0} |\pi_n^*(u) - e^u \sum_{k=0}^{r} \gamma_k(u) n^{-k}| du = O(n^{-(r+1)}) \qquad \text{a.s.}$$

where the functions $\gamma_k(u) = \gamma_k(u, \boldsymbol{x})$ are obtained successively from the relations $\alpha_k(u) = \sum_{j=0}^{k} \gamma_j(u)\beta_{k-j}$ and $\beta_k = \beta_k(\boldsymbol{x}) = \sum_{m=0}^{k}(-1)^{k-m}c_{k-m,m}(k+m)!$. This expansion of $\pi_n^*(u)$ is theoretically more appealing since it gives us an expansion in the powers of n^{-1}. However, the form presented in Theorem 3.1 is simpler for computational purposes.

Our proof is similar to that of Johnson (1970). Expanding in Taylor's series we have

$$\begin{aligned}
(3.1)\qquad \log Z_n(u) &= (u/\sigma_n)\sum_{i=1}^{n}(\partial/\partial\theta)\log f(X_i,\hat{\theta}_n) \\
&\quad + \frac{1}{2}(u/\sigma_n)^2\sum_{i=1}^{n}(\partial^2/\partial\theta^2)\log f(X_i,\hat{\theta}_n) + \cdots \\
&\quad + \frac{1}{(r+1)!}(u/\sigma_n)^{r+1}\sum_{i=1}^{n}(\partial^{r+1}/\partial\theta^{r+1})\log f(X_i,\hat{\theta}_n) \\
&\quad + \frac{1}{(r+2)!}(u/\sigma_n)^{r+2}\sum_{i=1}^{n}(\partial^{r+2}/\partial\theta^{r+2})\log f(X_i,\theta_n^*) \\
&= u + n\sum_{k=2}^{r+1} a_{kn}(\hat{\theta}_n)(u/\sigma_n)^k + n a_{r+2,n}(\theta_n^*)(u/\sigma_n)^{r+2},
\end{aligned}$$

where θ_n^* lies between $\hat{\theta}_n$ and $\hat{\theta}_n + u/\sigma_n$ and

$$(3.2)\qquad a_{kn}(\theta) = n^{-1}\sum_{i=1}^{n}(\partial^k/\partial\theta^k)\log f(X_i,\theta)/k!, \qquad k = 2,3,\ldots,r+2.$$

We denote the first $k+1$ terms of the Taylor expansion of $\pi(\theta)$ about $\hat{\theta}_n$ by $\pi_k(\theta)$, i.e.,

$$\pi_k(\theta) = \pi(\hat{\theta}_n) + (\theta-\hat{\theta}_n)\pi^{(1)}(\hat{\theta}_n) + \cdots + \frac{(\theta-\hat{\theta}_n)^k}{k!}\pi^{(k)}(\hat{\theta}_n),$$

where $\pi^{(j)}(\theta)$ denotes the j-th derivative of $\pi(\theta)$. Then

$$(3.3)\qquad \pi(\hat{\theta}_n + u/\sigma_n) = \pi_r(\hat{\theta}_n + u/\sigma_n) + (u/\sigma_n)^{r+1}\pi^{(r+1)}(\tilde{\theta}_n)/(r+1)!,$$

$\tilde{\theta}_n$ lying between $\hat{\theta}_n$ and $\hat{\theta}_n + u/\sigma_n$. We first prove the following lemma.

LEMMA 3.1. *There exists $\delta > 0$ such that with probability one*

$$\int_{-n\delta}^{0}\left|Z_n(u)\pi(\hat{\theta}_n + u/\sigma_n) - \exp\left\{u + n\sum_{k=2}^{r+1}a_{kn}(\hat{\theta}_n)(u/\sigma_n)^k\right\}\pi_r(\hat{\theta}_n + u/\sigma_n)\right|du \le M_1 n^{-(r+1)}$$

for some $M_1 > 0$ and for all sufficiently large n.

PROOF. The left hand side (LHS) of the desired inequality is dominated by

$$(3.4)\quad \int_{-n\delta}^{0} |\pi(\hat{\theta}_n + u/\sigma_n) - \pi_r(\hat{\theta}_n + u/\sigma_n)| Z_n(u) du \\ + \int_{-n\delta}^{0} |\pi_r(\hat{\theta}_n + u/\sigma_n)| \left| Z_n(u) - \exp\left\{ u + n \sum_{k=2}^{r+1} a_{kn}(\hat{\theta}_n)(u/\sigma_n)^k \right\} \right| du.$$

Since $\sigma_n/n \to c(\theta_0)$ a.s., by Lemma 2.1, with probability one for sufficiently small δ, the first term in (3.4) is dominated by

$$(3.5)\quad \int_{-n\delta}^{0} |(u/\sigma_n)^{r+1} \pi^{(r+1)}(\tilde{\theta}_n)/(r+1)!| Z_n(u) du \\ \le \frac{M_1'}{n^{r+1}(r+1)!} \int_{-\infty}^{0} |u|^{r+1} e^{u/2} du \\ = M_1''/n^{r+1},$$

where M_1', M_1'' are some constants. Using the inequality $|e^x - e^y| \le |x-y| e^x e^{|y-x|}$ and Lemma 1, we have the following upper bound for the second term in (3.4):

$$(3.6)\quad \int_{-n\delta}^{0} |\pi_r(\hat{\theta}_n + u/\sigma_n)| |n a_{r+2,n}(\theta_n^*)(u/\sigma_n)^{r+2}| e^{u/2} \\ \cdot \exp\{|n a_{r+2,n}(\theta_n^*)(u/\sigma_n)^{r+2}|\} du \\ \le n^{-(r+1)} B_1 \int_{-n\delta}^{0} |u|^{r+2} e^{u/2} \exp\{n^{-(r+1)} B_1 |u|^{r+2}\} du \\ \le n^{-(r+1)} B_1 \int_{-n\delta}^{0} |u|^{r+2} \exp(u/2 + B_1 \delta^{r+1} |u|) du \\ \le n^{-(r+1)} B_2.$$

Here we use (A6) and choose δ appropriately small; B_1, B_2 are constants. Combining (3.4), (3.5) and (3.6) we have the desired result. □

PROOF OF THEOREM 3.1. Setting $\sigma_n' = \sigma_n/n$ we have

$$(3.7)\quad \exp\left\{ u + n \sum_{k-2}^{r+1} a_{kn}(\hat{\theta}_n)(u/\sigma_n)^k \right\} \pi_r(\hat{\theta}_n + u/\sigma_n) \\ = e^u \left[\pi_r(\hat{\theta}_n + \sigma_n'^{-1}(u/n)) \exp\left\{ (u^2/n) \sum_{k-2}^{r+1} a_{kn}(\hat{\theta}_n)(u/n)^{k-2}/\sigma_n'^k \right\} \right].$$

For fixed n and sample sequence $\boldsymbol{x}$, the second factor on the RHS of (3.7) is a particular evaluation of the entire function

$$(3.8)\quad P_r(z, w, \boldsymbol{x}) = \pi_r(\hat{\theta}_n + z/\sigma_n') \exp\left\{ w \sum_{k=2}^{r+1} a_{kn}(\hat{\theta}_n) z^{k-2}/\sigma_n'^k \right\}$$

of two variables $z \in \mathbb{C}$ and $w \in \mathbb{C}$, where $\mathbb{C}$ is the set of all complex numbers. Thus we can write

$$P_r(z, w, \boldsymbol{x}) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} c_{lm}(\boldsymbol{x}) z^l w^m \tag{3.9}$$

where the series converges for all $z, w \in \mathbb{C}$. The coefficients $c_{lm} = c_{lm}(\boldsymbol{x})$ are given by

$$l!m!c_{lm} = (\partial^{l+m}/\partial^l z \partial^m w) P_r(z, w, \boldsymbol{x})|_{z=0, w=0}, \qquad l, m = 0, 1, 2, \ldots. \tag{3.10}$$

Collecting appropriate coefficients, it is easy to observe that $|c_{lm}(\boldsymbol{x})| \le M_{lm} < \infty$, where M_{lm}'s are constants. Using these estimates, for a given compact subset $\mathbb{K}$ of $\mathbb{C}^2$, we can find constants A_1 and A_2 such that

$$|P_r(z, w, \boldsymbol{x}) - \sum\sum_{l+m \le r} c_{lm}(\boldsymbol{x}) z^l w^m| \le A_1 |z|^{r+1} + A_2 |w|^{r+1} \quad \text{for} \quad (z, w) \in \mathbb{K}.$$

We denote the truncated series $\sum\sum_{l+m\le r} c_{lm}(\boldsymbol{x}) z^l w^m$ by $P_r^t(z, w, \boldsymbol{x})$. Thus for any $K > 0$, there exist $A_1, A_2 > 0$, such that

$$\begin{aligned} &\int_{-Kn^{1/2}}^{0} |\pi_r(\hat{\theta}_n + u/\sigma_n) \exp\left\{u + n \sum_{k=2}^{r+1} a_{kn}(\hat{\theta}_n)(u/\sigma_n)^k\right\} \\ &\qquad - e^u P_r^t(u/n, u^2/n, \boldsymbol{x})| du \\ &\quad = \int_{-Kn^{1/2}}^{0} e^u |P_r(u/n, u^2/n, \boldsymbol{x}) - P_r^t(u/n, u^2/n, \boldsymbol{x})| du \\ &\quad \le \int_{-Kn^{1/2}}^{0} e^u (A_1 |u/n|^{r+1} + A_2 |u^2/n|^{r+1}) du \\ &\quad \le M_2 n^{-(r+1)} \end{aligned} \tag{3.11}$$

for some constant $M_2 > 0$. From (3.11) and Lemma 3.1 it follows that there exists a constant $M_3 > 0$ such that with probability one,

$$\int_{-Kn^{1/2}}^{0} |\pi(\hat{\theta}_n + u/\sigma_n) Z_n(u) - e^u P_r^t(u/n, u^2/n, \boldsymbol{x})| du \le M_3 n^{-(r+1)}. \tag{3.12}$$

Now by Lemma 2.2, with probability one,

$$\begin{aligned} \int_{-\infty}^{-n\delta} \pi(\hat{\theta}_n + u/\sigma_n) Z_n(u) du &\le \int_{-\infty}^{-n\delta} \pi(\hat{\theta}_n + u/\sigma_n) e^{-n\varepsilon} du \\ &= O(n^{-(r+1)}). \end{aligned} \tag{3.13}$$

From Lemma 2.1, we also obtain

$$\begin{aligned}(3.14)\qquad &\int_{-n\delta}^{-Kn^{1/2}} \pi(\hat{\theta}_n + u/\sigma_n) Z_n(u)du \\ &\leq \exp[-Kn^{1/2}/2] \int_{-n\delta}^{-Kn^{1/2}} \pi(\hat{\theta}_n + u/\sigma_n)du \\ &= O(n^{-(r+1)}) \quad \text{a.s.}\end{aligned}$$

Also, it is easy to see that

$$(3.15)\qquad \int_{-\infty}^{-Kn^{1/2}} e^u P_r^t(u/n, u^2/n, \boldsymbol{x}) = O(n^{-(r+1)}) \quad \text{a.s.}$$

Combining (3.12)–(3.15) we have with probability one

$$(3.16)\qquad \int_{-\infty}^{0} |\pi(\hat{\theta}_n + u/\sigma_n) Z_n(u) - e^u P_r^t(u/n, u^2/n, \boldsymbol{x})|du \leq M_4 n^{-(r+1)},$$

for a constant $M_4 > 0$.

Set $C_n = \int_{-\infty}^{0} e^u P_r^t(u/n, u^2/n, \boldsymbol{x})du$ and $D_n = \int_{-\infty}^{0} \pi(\hat{\theta}_n + u/\sigma_n) Z_n(u)du$. From (3.15) we have

$$(3.17)\qquad |C_n - D_n| \leq M_4 n^{-(r+1)}.$$

Also putting $r = 0$ in (3.16) and noting that $P_0^t(u/n, u^2/n, \boldsymbol{x}) = \pi(\hat{\theta}_n)$, we have

$$(3.18)\qquad |D_n - \pi(\hat{\theta}_n)| \leq M_4 n^{-1}.$$

Now

$$\begin{aligned}&\int_{-\infty}^{0} |\pi_n^*(u \mid \boldsymbol{x}) - C_n^{-1} e^u P_r^t(u/n, u^2/n, \boldsymbol{x})|du \\ &\leq D_n^{-1} \int_{-\infty}^{0} |\pi(\hat{\theta}_n + u/\sigma_n) Z_n(u) - e^u P_r^t(u/n, u^2/n, \boldsymbol{x})|du \\ &\quad + \int_{-\infty}^{0} e^u P_r^t(u/n, u^2/n, \boldsymbol{x}) |C_n^{-1} - D_n^{-1}|du\end{aligned}$$

which, by (3.16), (3.17) and (3.18), is dominated by $Mn^{-(r+1)}$ for some constant M. Thus we have with probability one,

$$(3.19)\qquad \int_{-\infty}^{0} |\pi_n^*(u \mid \boldsymbol{x}) - C_n^{-1} e^u P_r^t(u/n, u^2/n, \boldsymbol{x})|du \leq Mn^{-(r+1)}.$$

From (3.9) we have

$$P_r^t(u/n, u^2/n, \boldsymbol{x}) = \sum_{l+m\le r} c_{lm} u^{l+2m} n^{-(l+m)} = \sum_{k=0}^{r}\left(\sum_{m=0}^{k} c_{k-m,m} u^{k+m}\right) n^{-k} \tag{3.20}$$

and hence

$$C_n = \sum_{l+m\le r} (-1)^l c_{lm}(l+2m)! n^{-(l+m)}. \tag{3.21}$$

Combining (3.19)–(3.21), we have the result. □

Below we give the expressions for the first few c_{lm}'s.

$$c_{00} = \pi(\hat\theta_n), \quad c_{01} = \pi(\hat\theta_n)a_{2n}(\hat\theta_n)/\sigma_n'^2, \quad c_{k0} = \pi^{(k)}(\hat\theta_n)/\sigma_n'^k, \quad 1\le k\le r,$$

$$c_{02} = \pi(\hat\theta_n)a_{2n}^2(\hat\theta_n)/(2\sigma_n'^4), \quad c_{11} = (\pi(\hat\theta_n)a_{3n}(\hat\theta_n) + \pi'(\hat\theta_n)a_{2n}(\hat\theta_n))\sigma_n'^{-3}.$$

From Theorem 3.1 we immediately have the following expansion for the posterior distribution function $F_n(u) = F_n(u, \boldsymbol{x}) = \int_{-\infty}^u \pi_n^*(t)dt$, $u \le 0$.

THEOREM 3.2. *Under the assumptions of Theorem* 3.1, *we have*

$$F_n(u) = C_n^{-1}\sum_{k=0}^{r} n^{-k}\beta_k(u)e^u + O(n^{-(r+1)}) \quad a.s.\ uniformly\ in\ u, \tag{3.22}$$

where $\beta_k(u) = \beta_k(u,\boldsymbol{x}) = \sum_{m=0}^k c_{k-m,m}Q_{k+m}(u)$ *and* $Q_s(u) = \sum_{l=0}^s (-1)^{s-l}u^l s!/l!$.

4. Expansions of moments and quantiles

In this section we study the moments and quantiles of the posterior distribution of $u = \sigma_n(\theta - \hat\theta_n)$. We first consider the posterior moments. We use the following result whose proof parallels that of Theorem 3.1.

THEOREM 4.1. *Let* $H(u)$, $u \le 0$, *be a nonnegative function such that*
(i) *for all* $\delta > 0$, $\int_{-\infty}^0 H(u)e^{\delta u}du < \infty$
(ii) *for some* $K_0 > 0$,

$$\lim_{n\to\infty} \exp[-K_0 n^{1/2}] \int_{-\infty}^{-K_0 n^{1/2}} H(u)\pi(\hat\theta_n + u/\sigma_n)du = 0 \quad a.s.$$

Then under the conditions of Theorem 3.1, *we have*
(a) $\int_{-\infty}^0 H(u)|\pi_n^*(u) - e^u C_n^{-1}\sum_{k=0}^r \alpha_k(u)n^{-k}|du = O(n^{-(r+1)})$ *a.s.*

(b) $\int_{-\infty}^{0} H(u)|\pi_n^*(u) - e^u \sum_{k=0}^{r} \gamma_k(u) n^{-k}| du = O(n^{-(r+1)})$ *a.s.*
where C_n, $\alpha_k(u)$, $\gamma_k(u)$, $k = 0, \ldots, r$, *are as defined in Theorem* 3.1 *and Remark* 3.1.

If $H(u) = |u|^s$, $s \geq 1$ is an integer, a sufficient condition for (ii) in Theorem 4.1 is given by

$$\int_{\Theta} |\theta|^s \pi(\theta) d\theta < \infty. \tag{4.1}$$

Thus if (4.1) is satisfied and (A1)–(A7) hold, from Theorem 4.1 we have

$$\int_{-\infty}^{0} u^s \pi_n^*(u) du - \int_{-\infty}^{0} u^s e^u C_n^{-1} \sum_{k=0}^{r} \alpha_k(u) n^{-k} du = O(n^{-(r+1)}) \quad \text{a.s.} \tag{4.2}$$

which yields

$$\int_{-\infty}^{0} u^s \pi_n^*(u) du = C_n^{-1} \sum_{k=0}^{r} \sum_{m=0}^{k} (-1)^{k+m+s} (k+m+s)! c_{k-m,m} n^{-k} + O(n^{-(r+1)}) \quad \text{a.s.} \tag{4.3}$$

Taking $s = 1$ in (4.3), we obtain the following asymptotic expansion for the posterior mean $\tilde{\theta}_n = \int_{\Theta} \theta \pi_n(\theta) d\theta$:

$$\tilde{\theta}_n = \hat{\theta}_n - (C_n \sigma_n)^{-1} \sum_{k=0}^{r} \sum_{m=0}^{k} (-1)^{k+m} c_{k-m,m} (k+m+1)! n^{-k} + O(n^{-(r+2)}) \quad \text{a.s.} \tag{4.4}$$

In particular, we get the following explicit one term expansion for the posterior mean by taking $r = 1$ in (4.4):

$$\begin{aligned} \tilde{\theta}_n = \hat{\theta}_n &- (\pi(\hat{\theta}_n)\sigma_n - \pi'(\hat{\theta}_n) + 2\sigma_n^{-1}\pi(\hat{\theta}_n) a_{2n}(\hat{\theta}_n))^{-1} \\ &\times (\pi(\hat{\theta}_n) - 2\sigma_n^{-1}\pi'(\hat{\theta}_n) + 6n\sigma_n^{-2}\pi(\hat{\theta}_n) a_{2n}(\hat{\theta}_n)) + O(n^{-3}) \quad \text{a.s.} \end{aligned}$$

Similar explicit two term expansion can be obtained using the values of c_{lm} given above. An asymptotic expansion for the posterior variance can also be obtained from (4.3) with $s = 2$.

We can also have the following results analogous to Theorems 4.1, 5.1 and 5.2 of Johnson (1970). Arguments involved are almost similar and hence the proofs are omitted.

THEOREM 4.2. *Let* $\eta_n(\xi) = \log F_n(\xi)$, $\xi < 0$. *Then under the assumptions of Theorem* 3.1, *there exist functions* $\omega_1(\xi), \omega_2(\xi), \ldots,$ *all polynomials in* ξ *with bounded coefficients, such that*

$$\eta_n(\xi) = \xi + \sum_{j=1}^{r} \omega_j(\xi) n^{-j} + O(n^{-(r+1)}) \quad \textit{a.s.},$$

uniformly in ξ belonging to compacts. The coefficients $\omega_j(\xi)$ may be obtained formally by identifying coefficients of n^{-j}, $j \geq 1$, in the identity $\sum_{k=0}^{r} \beta_k(\xi) n^{-k} = \exp[\sum_{k=0}^{r} \omega_k(\xi) n^{-k}]$, where $\beta_k(u)$'s are as defined in Theorem 3.2. (For example, $\omega_1(\xi) = \beta_1(\xi)$, $\omega_2(\xi) = \beta_2(\xi) - \omega_1^2(\xi)/2$, etc.)

THEOREM 4.3. *Let $\xi_n(\eta)$ be the solution of $F_n(\xi_n(\eta)) = e^\eta$, $\eta < 0$. Then under the assumptions of Theorem 3.1, we have the following expansion:*

$$\xi_n(\eta) = \eta + \sum_{j=1}^{r} \tau_j(\eta) n^{-j} + O(n^{-(r+1)}) \qquad a.s., \tag{4.5}$$

uniformly in η belonging to compacts. The coefficients $\tau_j(\eta) = \tau_j(\eta, \boldsymbol{x})$ are polynomials in η and can formally be obtained from the identity

$$\eta = \sum_{j=0}^{r} \tau_j(\eta) n^{-j} + \sum_{k=1}^{r} \omega_k \left(\sum_{j=0}^{r} \tau_j(\eta) n^{-j} \right) n^{-k}$$

by identifying the coefficients of n^{-k}, $k \geq 1$.

THEOREM 4.4. *Let $0 < \alpha < 1$ and $\eta = \log \alpha$. Then under the assumptions of Theorem 3.1, we have*

$$F_n \left(\eta + \sum_{j=1}^{r} \tau_j(\eta) n^{-j} \right) = \alpha + O(n^{-(r+1)}) \qquad a.s. \tag{4.6}$$

The expansions obtained in Theorems 4.2, 4.3 and 4.4 are of theoretical interest. It may be easier to find an approximation of the posterior p-th quantile by solving the equation $F_n^*(u) = p$ numerically where $F_n^*(u)$ is the approximation for the posterior distribution function $F_n(u)$ obtained from Theorem 3.2.

5. A simulation study

We now illustrate the usefulness of the theoretical results obtained above by means of a simulation experiment. For this we consider i.i.d. observations $X_1, \ldots, X_n$, from a uniform distribution over $[0, \theta]$ and a prior density $\pi(\theta) = e^{-\theta}$, $\theta > 0$. In this case $\hat{\theta}_n = X_{(n)}$, the maximum of the observations and $\sigma_n = -n/X_{(n)}$. The actual posterior distribution of $u = \sigma_n(\theta - \hat{\theta}_n) = n(1 - \theta/X_{(n)})$ is given by the density $\pi_n^*(u) = B_n^{-1} n^{-1} X_{(n)}^{-(n-1)} (1 - u/n)^{-n} \exp[-X_{(n)}(1 - u/n)]$ where $B_n = \int_{X_{(n)}}^{\infty} \theta^{-n} e^{-\theta} d\theta$.

From Theorem 3.1 (with a reparametrization $-\theta$) we obtain the expression for the one- and two-term expansions $\pi_{1n}^*(u)$ and $\pi_{2n}^*(u)$ of the posterior density by putting $r = 1$ and $r = 2$ respectively. In our experiment, i.i.d. observations are generated from a uniform distribution with $\theta = 1$. For different choices of

Table 1. Average L^1 distance of the posterior approximations from the exact posterior density based on 10 replications for different choices of the sample size n.

Sample size	Average distance (from the exact posterior) of		
	limit	one-term expansion	two-term expansion
$n = 5$	0.07585	0.05295	0.04291
$n = 10$	0.04625	0.01477	0.01094
$n = 15$	0.03254	0.00688	0.00499
$n = 25$	0.02008	0.00259	0.00187
$n = 50$	0.01050	0.00066	0.00046
$n = 100$	0.00531	0.00017	0.00012

the sample size n, we find the expressions for $\pi_n^*(u)$ and its approximations $\pi^*(u)$, $\pi_{1n}^*(u)$ and $\pi_{2n}^*(u)$ and also the corresponding posterior means, medians and variances of θ. For the expansions, posterior means and variances are computed using relations (4.3) and (4.4) while the medians are obtained by solving the equation $F_n^*(u) = 1/2$ numerically, $F_n^*(u)$ being the approximation of the posterior distribution function $F_n(u)$ obtained in Theorem 3.2. The whole experiment is replicated 10 times.

For comparison of the successive approximations π^*, π_{1n}^* and π_{2n}^* with the exact posterior density π_n^*, we compute the L^1 distances of each of the approximations from π_n^* for all the replications with different choices of n. It is observed that the approximations are pretty good even for a moderate sample size such as $n = 10$. Further, *in all the replications and for all the choices of n*, the approximations improve as we increase the order of expansions. We present in Table 1, the average (based on 10 replications) L^1-distances for different sample sizes. For computing the L^1-distance, we use Simpson's rule on the interval $[-10, 0]$ with grid length 10^{-4}. We also plot separately each of π^*, π_{1n}^* and π_{2n}^* together with π_n^*. For a typical replication with $n = 10$, these plots are shown in Figs. 1–3. For larger sample sizes like $n = 25$, the approximations are visually indistinguishable from the exact posterior.

Examining the values of the posterior means, medians and variances and the corresponding approximations (these are not presented here to save space), we find that the values obtained from the limit are generally underestimates of the actual values. There is a remarkable improvement in the one-term expansion which performs almost equally well as the two-term expansion in this regard.

In our experiment we have chosen the model $U[0, \theta]$ so that we are able to compute the exact posterior and compare it with its approximations. For more complicated models, the actual posterior may be awfully complicated while the approximations can be computed even without the help of a computer.

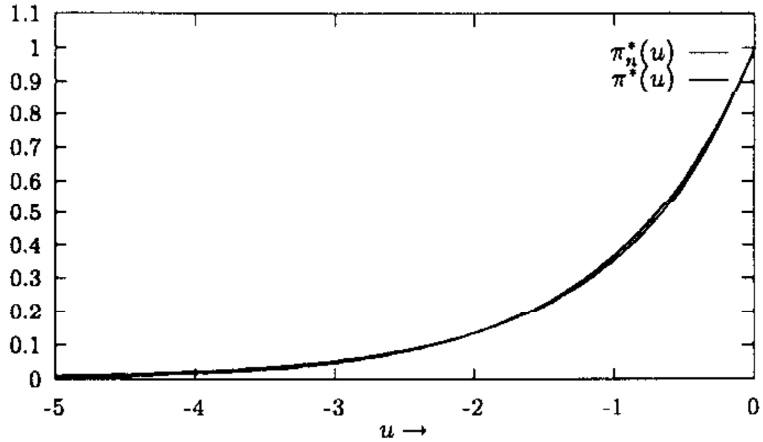

Fig. 1.

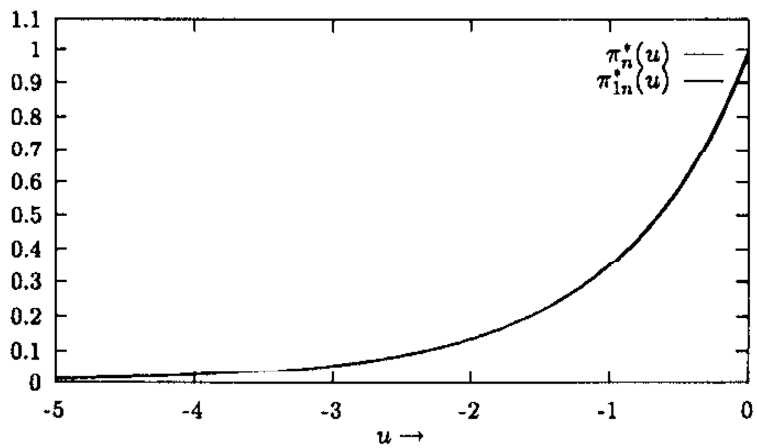

Fig. 2.

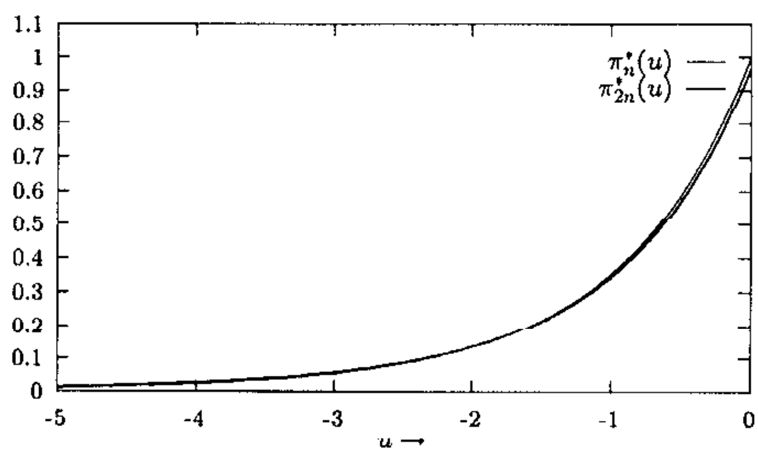

Fig. 3.

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