The exam is open book and open notes. No laptops or phones are permitted. You may use a calculator. Giving or receiving assistance from other students is not allowed. Each problem is worth 25 points. Show work to receive full credit! Partial credit will be given, but only for work written on the exam.

1. Assume a one-parameter model with parameter \( \theta \). The loss function comparing the estimator \( \hat{\theta}(y) \) and the true value \( \theta_0 \) is

\[
l[\hat{\theta}(y), \theta_0] = [\hat{\theta}(y) - \theta_0]^2 + \lambda [\hat{\theta}(y) - c]^2,
\]

where \( \lambda > 0 \) and \( c \) are a fixed constants.

(a) Give a real-world scenario where this loss might be used.

(b) What is the Bayes rule with respect to this loss?

\[
\text{risk} = E_{\theta|y} \left( l(\theta, \theta_0) \right) = E_{\theta|y} \left( (\hat{\theta} - \bar{\theta} + \bar{\theta} - \theta_0)^2 + \lambda (\hat{\theta} - c)^2 \right) \\
= E_{\theta|y} (\hat{\theta} - \bar{\theta})^2 + \lambda (\hat{\theta} - c)^2
\]

\[
\text{risk}' = 2 (\hat{\theta} - \bar{\theta}) + 2 \lambda (\hat{\theta} - c)
\]

\[
\text{so} \quad \hat{\theta} = \frac{\bar{\theta} + \lambda c}{1 + \lambda}
\]

is the Bayes rule.

\[
\text{risk}'' = 2 + 2 \lambda > 0.
\]
Say $Y | \mu, \sigma, \xi \sim \text{Fréchet}(\mu, \sigma, \xi)$ with density function

$$f(y|\mu, \sigma, \xi) = \frac{\xi}{\sigma} \left( \frac{y - \mu}{\sigma} \right)^{-1-\xi} \exp \left[ \left( \frac{y - \mu}{\sigma} \right)^{-\xi} \right] I(y > \mu),$$

where $\mu \in \mathbb{R}$ is the location parameter, $\sigma > 0$ is the scale parameter, and $\xi > 0$ is the shape parameter. Identify a value of $(\mu, \xi)$ so that when these parameters are fixed at this value, $\sigma$ has a conjugate prior. Specify the conjugate prior in the case and give the resulting posterior (assuming a single observation for the likelihood).

Say $\mu = 0$ and $\xi = 1$. Then

$$f(y|\sigma) = \frac{1}{\sigma} \frac{y^{-2}}{e^{-y/\sigma}} e^{-\frac{y}{\sigma}} = \sigma y^{-2} e^{-\sigma y}.$$

So $\sigma \sim \text{Gamma}(s, b)$ is conjugate. The posterior is

$$p(\sigma | y) \propto p(y | \sigma) p(\sigma)$$

$$\propto \sigma^{-s} e^{-\sigma(y^{-1} + b)}$$

$$\propto \sigma^{a-1} e^{-\sigma(y^{-1} + b)}$$

and thus $\sigma | y \sim \text{Gamma}(a+1, b+\frac{1}{y})$. 

3. An ecologist scans a forest for a particular species of frog \( n \) consecutive days, and observes the species \( Y \in \{0, \ldots, n\} \) times. To model the data, she assumes likelihood \( Y|\theta, Z \sim \text{Binomial}(n, \theta Z) \) and priors \( \theta \sim \text{Unif}(0, 1) \) and \( Z \sim \text{Bernoulli}(0.5) \). \( Z \in \{0, 1\} \) indicates whether the species occupies the forest. If \( Z = 0 \) the forest is not occupied and \( Y = 0 \) with probability one; if \( Z = 1 \), the forest is occupied and each day the species is detected with probability \( \theta \in (0, 1) \) so \( Y|\theta, Z = 1 \sim \text{Binomial}(n, \theta) \).

(a) What is the posterior probability that the forest is occupied, \( \text{Prob}(Z = 1|Y) \)?

\[
P(Y = 0) = \int_0^1 p(Y = 0, \theta | Y = 0) d\theta
\]

\[
P(Y = 0) = \int_0^1 \frac{p(Y = 0, \theta | Y = 0) p(\theta)}{p(Y)} d\theta = \int_0^1 \frac{1 - \frac{1}{2} \theta p(\theta)}{p(Y)} d\theta = \frac{1}{2 p(Y)}
\]

\[
P(Y = 0) = \int_0^1 \frac{p(Y = 0, \theta | Y = 0) \theta}{p(Y)} d\theta = \frac{1}{2 p(Y)} \int_0^1 \theta (1 - \theta)^n d\theta = \frac{1}{2 p(Y)} \int_0^1 \theta (1 - \theta)^{n+1-1} d\theta
\]

\[
P(Y = 0) = \frac{1}{2 p(Y)} \frac{p(n+1) p(1)}{p(n+2)} = \frac{1}{2 p(Y) (n+2)}
\]

\[
P(Y = 0) = \frac{p(2 = 1 | Y = 0)}{p(2 = 1 | Y = 0) + p(2 = 0 | Y = 0)} = \frac{\frac{1}{n+1}}{\frac{1}{n+1} + 1} = \frac{1}{n+2}
\]

(b) How many times would she have to sample before \( \text{Prob}(Z = 1|Y = 0) < 0.05 \)?

Pick the smallest \( n \) so that \( \frac{1}{n+2} < \frac{1}{2 \theta} \).

\[
\theta = 0.05
\]

\[
n = 18
\]
4. Let $Y|\theta, \pi \sim \text{Normal}(\pi \theta, 1)$ where $\theta \in \mathcal{R}$.

(a) Assuming $\pi$ is fixed, what is the Jefferys’ prior for $\theta$?

\[
\begin{align*}
\mathcal{L} &= \text{const} - \frac{1}{2} (Y - \pi \theta)^2 \\
\mathcal{L}' &= \pi (Y - \pi \theta) \\
\mathcal{L}'' &= -\pi^2 \\
E(\mathcal{L}'') &= -\pi^2
\end{align*}
\]

so $\int \mathcal{L}(\theta) \propto 1$ and so it is a flat prior.

(b) Let $\pi \sim \text{Bernoulli}(p)$ for $p \in (0, 1)$, and assume that $\pi$ and $\theta$ have independent priors. Does the improper prior $p(\theta) \propto 1$ give a proper posterior? Justify your answer.

\[
\begin{align*}
N011 \quad &\int p(\theta|Y, \pi = 0) \propto \int f(Y|\theta, \pi = 0) p(\theta) d\theta \\
&\propto \int p(\theta) d\theta = \infty.
\end{align*}
\]

(c) Now assume $\pi \sim \text{Bernoulli}(p)$ and $\theta \sim \text{N}(0, \tau^2)$. To test the hypothesis that the mean of $Y$ is zero, we often compute

\[
P(\pi = 0|Y = y) = \frac{(1 - p)\tau}{\sqrt{\tau p \exp(-\frac{\tau y^2}{2}) + (1 - p)\tau}}
\]

where $\tau = \tau^2/(1 + \tau^2)$. What is the limit of $P(\pi = 0|Y = y)$ as $\tau^2 \to \infty$? Why might this be problematic?

\[
\lim_{\tau \to \infty} P(\pi = 0|Y = y) \to 1. \text{ This is a problem because it indicates that for flat priors the data play no role.}
\]