Common one-parameter models

In this section we will explore common one-parameter models, including:

1. Binomial data with beta prior on the probability

2. Poisson data with gamma prior on the rate

3. Gaussian data with fixed variance and normal prior on the mean

4. Gaussian data with fixed mean and inverse gamma prior on the variance

As we go through these examples, note the:

1. Relative contribution the data and prior to the posterior

2. Effect of the prior as the sample size increases

3. Differences between posteriors and MLEs
Beta/Binomial

Say \( Y|\theta \sim \text{Binomial}(n, \theta) \) and \( \theta \sim \text{Beta}(a, b) \), compute the posterior of the \( \theta \).

Therefore \( a \) and \( b \) can be interpreted as the “prior number of successes and failures”, respectively.
Beta/Binomial

1. The prior mean and variance are $E(\theta) = \frac{a}{a+b}$ and $V(\theta) = \frac{ab}{(a+b)^2(a+b+1)}$.

2. The MLE is $\hat{\theta}_{MLE} = Y/n$.

3. The posterior mean and variance are:

4. Which $a$ and $b$ have the posterior mean and MLE agree?

5. What are $\lim_{n \to \infty} E(\theta|Y)$ and $\lim_{n \to \infty} V(\theta|Y)$? Interpret these results.
Beta/Binomial

beta_binom<-function(n,y,a=1,b=1,main=""){
  #likelihood: y|theta~binom(n,theta)
  #prior: theta~beta(a,b)
  #posterior: theta|y~beta(a+y,n-y+b)

  theta<-seq(0.001,0.999,0.001)
  prior<-dbeta(theta,a,b)
  if(n>0){likelihood<-dbinom(rep(y,length(theta)),n,theta)}
  if(n>0){posterior<-dbeta(theta,a+y,n-y+b)}

  #standardize!
  prior<-prior/sum(prior)
  if(n>0){likelihood<-likelihood/sum(likelihood)}
  if(n>0){posterior<-posterior/sum(posterior)}

  ylim<-c(0,max(prior))
  if(n>0){ylim<-c(0,max(c(prior,likelihood,posterior)))}

  plot(theta,prior,type="l",lty=2,xlab="theta",ylab="",main=main,ylim=ylim)
  if(n>0){lines(theta,likelihood,lty=3)}
  if(n>0){lines(theta,posterior,lty=1,lwd=2)}
  legend("topleft",c("prior","likelihood","posterior"),
          lty=c(2,3,1),lwd=c(1,1,2),inset=0.01,cex=.5)
}

Code is online at http://www4.stat.ncsu.edu/~reich/ST740/code/beta_binom.R.
par(mfrow=c(2,2))
beta_binom(3,2,2.5,7.5,main="Prior: beta(2.5,7.5), data: 2/3")
beta_binom(3,2,25,75,main="Prior: beta(25,75), data: 2/3")
beta_binom(30,20,2.5,7.5,main="Prior: beta(2.5,7.8), data: 20/30")
beta_binom(300,200,2.5,7.5,main="Prior: beta(2.5,7.5), data: 200/300")
Poisson/Gamma

Say we monitor a patient for $N$ days and observe $Y$ seizures. Our goal is to estimate the seizure rate (expected number per day) $\theta$. Our model is $Y|\theta \sim \text{Poisson}(N\theta)$ and $\theta \sim \text{Gamma}(a, b)$ with density $p(\theta) \propto \theta^{a-1} \exp(-b\theta)$. Compute the posterior of the $\theta$. 
Poisson/Gamma

1. The MLE is

2. The posterior mean is

3. Interpret the roles of $a$ and $b$.

4. Suggest an “uninformative prior”.

5. Suggest a way to build an “informative prior”.

Normal with fixed variance and unknown mean

Say $Y_1, \ldots, Y_n \sim N(\mu, \sigma^2)$ and $\mu \sim N(\theta, \tau^2)$, find the posterior of the $\mu$. 
Normal with fixed variance and unknown mean

After some algebra, we find $V(\mu|y) = \frac{\sigma^2 + \sum_{j=1}^{n}y_j^2}{\sigma^2 + n\tau^2}$.

1. What happens as $n \to \infty$? Why is this reasonable?

2. What happens as $\tau \to \infty$? Why is this reasonable?

After some algebra, we find $E(\mu|y) = \frac{nr}{nr+1} \bar{Y} + \frac{1}{nr+1} \theta$ where $r = \frac{\tau^2}{\sigma^2}$.

1. Explain how the posterior mean combines the prior and posterior.

2. What happens as $r \to 0$? Why is this reasonable?

3. What happens as $r \to \infty$? Why is this reasonable?

4. What happens as $n \to \infty$? Why is this reasonable?
Normal with fixed mean and unknown variance

Say $Y_1, \ldots, Y_n \sim N(\mu, \sigma^2)$ and $\sigma^2 \sim \text{InvGamma}(a, b)$ with density $(\sigma^2)^{-a-1} \exp\left(-\frac{b}{\sigma^2}\right)$ and mean (for $a > 1$) $b/(a + 1)$. Find, the posterior of $\sigma^2$. 
Normal with fixed mean and unknown variance

1. The MLE is

2. The posterior mean is

3. Interpret the roles of $a$ and $b$.

4. Suggest an “uninformative prior”.

5. Suggest a way to build an “informative prior”.