Bayes’ Theorem

- In Bayesian statistics, we select the prior, \( p(\theta) \), and the likelihood, \( p(y|\theta) \).

- Based on these two pieces of information, we must compute the posterior \( p(\theta|y) \).

- Bayes’ theorem is the mathematical formula to convert the likelihood and prior to the posterior.

- Bayes theorem:

\[
p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}
\]

Therefore the posterior is proportional to the likelihood times the prior.
Derivation Bayes' Theorem

Have class do this

By definition

1. \( p(y|\theta) = \frac{p(y, \theta)}{p(\theta)} \)

2. \( p(\theta|x) = \frac{p(y, \theta)}{p(y)} \)

From 1, \( p(y, \theta) = p(y|\theta)p(\theta) \). Inserting this into 2 gives

\[ p(\theta|x) = \frac{p(y|\theta)p(\theta)}{p(y)} \]
Example from probability

A team plays half its games at home, wins 70% of its home games, and 40% of its road games. Give that the team wins a game, what’s the probability it was a home game?

\[ Y = \begin{cases} 1 & \text{win} \\ 0 & \text{loss} \end{cases} \quad X = \begin{cases} 1 & \text{home} \\ 0 & \text{away} \end{cases} \]

\( \begin{align*}
(1) & \Rightarrow P(X=0) = P(X=1) = \frac{1}{2} \\
(2) & \Rightarrow p(Y=1 | X=1) = 0.7 \\
(3) & \Rightarrow p(Y=1 | X=0) = 0.3
\end{align*} \)

We want \( p(X=1 | Y=1) \).

By Bayes' rule

\[
p(X=1 | Y=1) = \frac{p(Y=1 | X=1) p(X=1)}{p(Y=1)}
\]

\[
= \frac{0.7 \cdot 0.5}{0.55}
\]

What's the marginal prob of winning?

\[
p(Y=1) = \frac{1}{2} \cdot 0.7 + \frac{1}{2} \cdot 0.4 = 0.55
\]

So

\[
p(X=1 | Y=1) = \frac{0.7 \cdot 0.5}{0.55} = \frac{7}{11}
\]
HIV example

- Let $\theta$ be the parameter of interest with

$$\theta = \begin{cases} 
0 & \text{patient does not have HIV} \\
1 & \text{patient has HIV.} 
\end{cases}$$

- The data is $Y$, defined as

$$Y = \begin{cases} 
0 & \text{test is negative} \\
1 & \text{test is positive.} 
\end{cases}$$

- **Likelihood:** Since $Y$ is binary, we use a Bernoulli PMF for the likelihood:

  \begin{align*}
  &\text{If } \theta = 0, \quad Y \sim \text{Bern}(\theta) \\
  &\quad \Pr(Y = 0|\theta = 0) = \theta \quad \Pr(Y = 1|\theta = 0) = 1 - \theta \\
  &\text{If } \theta = 1, \quad Y \sim \text{Bern}(\theta, \lambda) \\
  &\quad \Pr(Y = 0|\theta = 1) = \lambda \quad \Pr(Y = 1|\theta = 1) = 1 - \lambda
  \end{align*}

- We must specify both $\Pr(Y = 1|\theta = 0) = q_0$ and $\Pr(Y = 1|\theta = 1) = q_1$.

- The false positive rate is $q_0$; the true positive rate is $q_1$. How might we select $q_0$ and $q_1$?

- **Prior:** Since $\theta$ is binary, we use a Bernoulli prior with $\Pr(\theta = 1) = p$. The prior PMF is

$$\Pr(\theta = x) = \binom{x}{k} p^k (1-p)^{1-k} = \begin{cases} 
p & x = 0 \\
1 - p & x = 1
\end{cases}$$

- $p$ is our guess about the probability the patient has HIV before taking the test. How might we select $p$?
HIV example

Given the patient tests positive, what is the probability they have HIV?

In math, the question is \( P(\theta = 1 \mid y = 1) \).

Bayes rules say:

\[
P(\theta = 1 \mid y = 1) = \frac{P(y = 1 \mid \theta = 1) P(\theta = 1)}{P(y = 1)}
\]

\[
= \frac{\varepsilon_1 p}{\varepsilon_1 p + \varepsilon_0 (1-p)}
\]

We also know that:

\[
P(\theta = 0 \mid y = 1) = \frac{P(y = 1 \mid \theta = 0) P(\theta = 0)}{P(y = 1)}
\]

\[
= \frac{\varepsilon_0 (1-p)}{\varepsilon_1 p + \varepsilon_0 (1-p)}
\]

Do we need to compute \( P(y = 1) \)? No! We know \( P(\theta = 0 \mid y = 1) + P(\theta = 1 \mid y = 1) = 1 \), so

\[
\frac{\varepsilon_1 p}{P(y = 1)} + \frac{\varepsilon_0 (1-p)}{P(y = 1)} = 1 \implies P(y = 1) = \frac{\varepsilon_1 p + \varepsilon_0 (1-p)}{1}
\]
HIV example

Given the patient tests negative, what is the probability they have HIV?

Following similar steps

\[ P(\theta = 1 | y = 0) \]

\[ = \frac{\theta (1-\theta_1)}{(1-\theta_1) \theta + (1-\theta_0) (1-\theta)} \]

| $\theta_0$ | $\theta_1$ | $\theta$ | Your guess | $P(\theta = 1 | y = 1)$ |
|-----------|-----------|---------|------------|-----------------|
| 0.5       | 0.5       | 0.2     |            |                 |
| 0         | 0.5       | 0.2     |            |                 |
| 0.01      | 0.99      | 0.2     | 0.001      |                 |
| 0.01      | 0.99      | 0.2     | 0.001      |                 |
| 0.01      | 1.00      | 0.2     |            |                 |

These probabilities are illustrated online at

http://www4.stat.ncsu.edu/~reich/st590/code/HIV.html