Solution for weekly review exercises #9

Prepared by Chen-Yen Lin

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(a) First argue that $X_{(n)}$ is complete sufficient for $\theta$.

i. Since $\frac{X_{(n)}}{\theta} \sim Beta(n, 1)$, it follows that $E X_{(n)} = \frac{\theta \cdot n}{n+1}$. Thus, $\frac{n+1}{n} X_{(n)}$ is the UMVUE of $\theta$.

ii. Similarly, $E\left(\frac{X_{(n)}}{\theta}\right)^2 = \frac{n}{n+2}$. Thus, $\frac{n+2}{n} X_{(n)}^2$ is the UMVUE of $\theta^2$.

iii. The reciprocal of $\frac{X_{(n)}}{\theta}$ has expectation $\frac{n}{n-1}$. Thus, $\frac{n-1}{n} \frac{1}{X_{(n)}}$ is the UMVUE of $1/\theta$.

(b) i. The Poisson density can be written as

$$f(x_i|\lambda) = \frac{1}{x_i!} e^{-\lambda} \exp \left[ \log \lambda x_i \right]$$

Since the density function belongs to an exponential family and the parameter space contains an open set, $T = \sum_{i=1}^n X_i$ is a complete sufficient statistic.

ii. One can show that the MLE of $\lambda$ is $\bar{X}$. By invariance property of MLE, the MLE of $\lambda^2$ is $\bar{X}^2$.

iii. $P(X_1 = x_1|T = t) = \frac{P(X_1 = x_1, T = t)}{P(T = t)} = \frac{P(X_1 = x_1, \sum_{i=2}^n X_i = t - x_1)}{P(T = t)}$

$$= \frac{e^{-\lambda} \frac{x_1!}{x_1!} e^{-(n-1)\lambda} \frac{(n-1)!}{(t-x_1)!} \frac{(t-x_1)!}{(n-1)!} e^{-n\lambda (n\lambda)!}}{t!} \frac{1}{x_1!(t-x_1)!} \left( \frac{1}{n} \right)^{x_1} \left( 1 - \frac{1}{n} \right)^{t-x_1}$$

That is, $X_1|T = t \sim Bin(t, n^{-1})$

iv. First show unbiasedness

$$E(X_1^2 - X_1) = \lambda + \lambda^2 - \lambda = \lambda^2$$

By Rao-Blackwell, we can improve $X_1^2 - X_1$ by taking conditional expectation of $X_1^2 - X_1$ given $T$. That is,

$$E(X_1^2 - X_1|T) = T \frac{1}{n} \left( 1 - \frac{1}{n} \right) + \left( \frac{T}{n} \right)^2 - \frac{T}{n}$$
An unbiased estimator of $e^{-\lambda}$ is $I(X_1 = 0)$. By using Rao-Blackwell method, the UMVUE of $e^{-\lambda}$ can be found by deriving the conditional expectation $E(I(X_1 = 0)|T)$

$$E(I(X_1 = 0)|T) = P(X_1 = 0|T = t) = \left(1 - \frac{1}{n}\right)^t$$

(c) All of the following questions have the form $E(T_1|T_2)$ where $T_1$ consists only one random variable and $T_2$ is a complete sufficient statistic. Thus, the conditional expectation is actually the UMVUE of $ET_1$.

(a) First realize that $EX_1 = \mu$, we have $E(X_1|\bar{X}) = \bar{X}$ since $\bar{X}$ is the UMVUE of $\mu$.

(b) $E(X_n^2|\sum_{i=1}^n X_i^2) = \frac{\sum_{i=1}^n X_i^2}{n}$

(c) $E(2X_1|X_{(n)}) = \frac{n+1}{n}X_{(n)}$

(d) $E(nX_{(1)}|\sum_{i=1}^n X_i) = \frac{\sum_{i=1}^n X_i}{n}$