(a)  

i. First recall that $EX_1 = \theta + \gamma$, where $\gamma$ is the Euler constant. Thus, by method of moment, we solve

$$n^{-1} \sum_{i=1}^{n} x_i = \theta + \gamma, \quad \Rightarrow \hat{\theta}_{MOM} = \bar{X} - \gamma$$

Secondly, the log-likelihood function can be written as

$$\log L(\theta) = -\sum_{i=1}^{n} x_i + n\theta - e^\theta \sum_{i=1}^{n} e^{-x_i}$$

By taking partial derivative w.r.t. $\theta$ and set to zero, the MLE is given by

$$\frac{\partial \log L}{\partial \theta} = n - e^\theta \sum_{i=1}^{n} e^{-x_i}, \quad \Rightarrow \hat{\theta}_{MLE} = \log \left( \frac{n}{\sum_{i=1}^{n} e^{-x_i}} \right)$$

Since the second-order derivative is negative, $\hat{\theta}_{MLE}$ indeed maximizes the log-likelihood function.

ii. Since the distribution is symmetric about $\theta$, it follows that $EX_1 = \theta$, then the moment estimator of $\theta$ is $\bar{X}$. Secondly, the log-likelihood function is given by

$$\log L(\theta) = -n \log 2 - \sum_{i=1}^{n} |x_i - \theta|$$

To maximize the log-likelihood function, it is equivalent to minimize the second absolute loss. From a population perspective, when $\theta$ is chosen to be population median, the absolute loss will be minimized. By using the analogy, $\hat{\theta}_{MLE} = \text{med}(X_1, \ldots, X_n)$

iii. As double exponential case, this distribution is symmetric about $\theta$, and therefore the moment estimator of $\theta$ is $\bar{X}$. Secondly, the log-likelihood function is

$$\log L(\theta) = n\theta - n\bar{x} - 2\sum_{i=1}^{n} \log[1 + \exp(-(x_i - \theta))]$$

By differentiating log-likelihood w.r.t. $\theta$ and set to zero, we need to solve

$$l' = n - 2\sum_{i=1}^{n} \frac{\exp[-(x_i - \theta)]}{1 + \exp[-(x_i - \theta)]} = 0$$

The difficulty lies on there is no analytic solution for $\hat{\theta}_{MLE}$. But by observing that $\lim_{\theta \to \infty} l'(\theta) = -n$ and $\lim_{\theta \to \infty} l'(\theta) = n$, there must exists some value on the real line such that $l' = 0$. One might need numerical solution to get an estimate.
(b) i. The log-likelihood function in this case is

$$\log L(\theta) = -n\lambda + \sum_{i=1}^{n} x_i \log \lambda - \log \sum_{i=1}^{n} x_i!$$

By setting the partial derivative to zero, we can solve the MLE of $\theta$ as

$$-n + \frac{\sum_{i=1}^{n} x_i}{\theta} = 0, \quad \Rightarrow \hat{\theta}_{MLE} = \bar{X}$$

The second-order derivative of the log-likelihood function is always negative regardless what value of $\theta$, thus $\hat{\theta}$ maximizes the likelihood function.

To argue consistency, it can be shown by WLLN since $X_i$ are random sample from a population whose second moment exists. Thus, $\bar{X} \xrightarrow{p} \theta$.

ii. First show $\tilde{\theta}$ is also unbiased for $\theta$.

$$E\tilde{\theta} = \frac{1}{n-1} \left[ E\sum_{i=1}^{n} X_i^2 - nE\bar{X}^2 \right] = \frac{1}{n-1} \left[ n(\theta + \theta^2) - n \left( \frac{\theta}{n} + \theta^2 \right) \right] = \theta$$

Secondly, we know $\tilde{\theta} = (n - 1)^{-1} \sum_{i=1}^{n} X_i^2 - \frac{n}{n-1} \bar{X}^2$. Again, by WLLN, the first term converges in probability to $\theta^2 + \theta$ and the second term converges in probability to $\theta^2$. Thus, $\tilde{\theta}$ is also a consistent estimator of $\theta$.

(c) i. First realize that $E\hat{\sigma}^2 = \sigma^2$ and $\sqrt{\hat{\sigma}^2}$ is a concave function. By Jensen’s inequality, $\sqrt{E\hat{\sigma}^2} = \sigma \geq E\sqrt{\hat{\sigma}^2}$. Thus, $\hat{\sigma}$ is a biased estimator. By CLT, we can find an asymptotic distribution for $\hat{\sigma}$, that is $\sqrt{n}(\hat{\sigma} - \sigma) \xrightarrow{d} N(0, 3\sigma^4)$. Furthermore, by Delta method, the asymptotic distribution for $\hat{\sigma}$ is given by $\sqrt{n}(\hat{\sigma} - \sigma) \xrightarrow{d} N(0, \frac{3\sigma^4}{n})$.

ii. Let $Y_i = |X_i|$, then $Y_i$ has a folded normal density. Next we show $\tilde{\sigma}$ is unbiased

$$E\tilde{\sigma} = \sqrt{\frac{\pi}{2}} E|X_1| = \sqrt{\frac{\pi}{2}} \int_{0}^{\infty} y \frac{2}{\pi \sigma^2} e^{-\frac{y^2}{2\sigma^2}} dy = \frac{1}{\sigma} \int_{0}^{\infty} ye^{-\frac{y^2}{2\sigma^2}} dy = \sigma$$

The variance of $\tilde{\sigma}$ can be found as follows

$$\text{Var}(\tilde{\sigma}) = \frac{\pi}{2} \text{Var}(|X_1|) = \frac{\pi}{2n} \left\{ E|X_1|^2 - (E|X_1|)^2 \right\} = \frac{\pi}{2n} \left( \sigma^2 - \frac{2}{\pi} \sigma^2 \right) = \frac{\sigma^2}{n} \left( \frac{\pi}{2} - 1 \right)$$