(5.44)(e)

\[
\sqrt{n}(S^2 - p(1-p)) = \sqrt{n}(S^2 - Y_n(1 - Y_n) + Y_n(1 - Y_n) - p(1-p)) \\
= \sqrt{n} \left( \frac{n}{n-1}Y_n(1 - Y_n) - Y_n(1 - Y) \right) + \sqrt{n}[Y_n(1 - Y_n) - p(1-p)] \\
= \frac{\sqrt{n}}{n-1}Y_n(1 - Y_n) + \sqrt{n}[Y_n(1 - Y_n) - p(1-p)]
\]

Since \(Y_n(1 - Y_n) \xrightarrow{p} p(1-p)\) and \(\frac{\sqrt{n}}{n-1} \to 0\), the first term in the last equation converges in probability to 0 whereas the second term converges in distribution to \(N(0, (1-2p)^2p(1-p))\) from (b). Thus, by Slutsky Theorem, their sum converges in distribution to \(N(0, (1-2p)^2p(1-p))\).

(a) By conditioning on \(Y = y\), the conditional distribution of \(X|Y = y\) becomes

\[
P(X|Y = y) = \begin{cases} 
\frac{1}{2}, & \text{if } X = y, \text{ or } X = -y 
\end{cases}
\]

Since the conditional distribution does not depend on \(\theta\), \(Y = |X|\) is sufficient for \(\theta\). Moreover, suppose \(Eg(Y) = 0\) for all \(\theta\), we have

\[
Eg(Y) = \int_0^\theta g(y) \frac{1}{\theta} dy = 0 \Rightarrow g(\theta) \frac{1}{\theta} = 0
\]

Since \(\theta > 0\), we have \(g(y) = 0\) for all \(y\). Therefore, \(Y = |X|\) is complete.

It can be shown that \(Z\) is a Bernoulli random variable with success probability 0.5. Since the distribution of \(Z\) does not depend on \(\theta\), it is an ancillary statistic and therefore independent of \(Y\) by Basu Theorem.

(b) We can define \(X_i = Z_i + \theta\), where \(Z_i \sim \exp(1)\). By realizing \(Z_{(1)}\) is also an exponential distribution with expectation \(1/n\), then \(X_{(1)}\) has density

\[
X_{(1)} \sim f(x_1) = n \exp[-n(x - \theta)], \quad x_1 > \theta
\]

Suppose \(Eg(X_{(1)}) = 0\) for all \(\theta\), we have

\[
Eg(X_{(1)}) = \int_0^\infty g(x_1)n \exp[-n(x_1 - \theta)]dx_1 = ne^{n\theta} \int_0^\infty g(x_1)e^{-nx_1}dx_1 = 0
\]

\[
\Rightarrow \frac{\partial Eg(X_{(1)})}{\partial \theta} = ne^{n\theta}(-1)g(\theta)e^{-n\theta} = -ng(\theta) = 0
\]

As a result, we establish \(g(x_1) = 0\), and therefore \(X_{(1)}\) is complete. Secondly, \(X_{(n)} - X_{(1)} = Z_{(n)} + \theta - Z_{(1)} - \theta = Z_{(n)} - Z_{(1)}\) whose distribution does not depend on \(\theta\). Thus, by Basu Theorem, \(X_{(1)}\) is independent of \(X_{(n)} - X_{(1)}\).
(c)  
(i) The density for $X_i$ can be found by using change-of-variable technique, which is given by

$$f(x_i|\theta) = e^{-x_i}e^\theta \exp[-e^\theta e^{-x_i}]$$

It is clear that the density belongs to an exponential family. By Theorem 6.2.25, $T(X) = \sum_{i=1}^n e^{-X_i}$ is a complete sufficient statistic.

(ii) The joint density of $X_1, \ldots, X_n$ is given by

$$f(x|\sigma) = \exp\left[\frac{1}{\sigma} \sum_{i=1}^n x_i - \sum_{i=1}^n e^{-\frac{x_i}{\sigma}}\right]$$

It is clear that there are only two trivial sufficient statistics $T_1(X) = \{X_1, \ldots, X_n\}$ and $T_2(X) = \{X_{(1)} < X_{(2)} < \ldots < X_{(n)}\}$. We claim that none of them is complete since we can find two non-zero functions $g_1(T_1)$ and $g_2(T_2)$ whose expectations are zero. More specifically, $g_1(T_1) = X_1 - \bar{X}$ and $g_2(T_2) = X_{(n)} - (1 + \frac{\log n}{\gamma})n^{-1} \sum_{i=1}^n X_{(i)}$, where $\gamma$ is the Euler constant. Before we proceed to show $g_1$ and $g_2$ has expectation zero, it would be easier to know the relation between extreme value distribution and exponential distribution, namely, if $W$ is a standard exponential random variable, then $Z \sim -\log W$. Then we can show the expectation of $g_1$ and $g_2$ are both zero.

$$Eg_1(T_1) = EX_1 - E\bar{X} = EX_1 - n^{-1} \sum_{i=1}^n EX_1 = 0$$

$$Eg_2(T_2) = EX_{(n)} - \left(1 + \frac{\log n}{\gamma}\right) n^{-1} \sum_{i=1}^n EX_{(i)}$$

$$= \sigma EZ_{(n)} - \left(1 + \frac{\log n}{\gamma}\right) E\bar{X}$$

$$= \sigma E(-\log W_{(1)}) - \left(1 + \frac{\log n}{\gamma}\right) EX_1$$

$$= \sigma \int_0^\infty -\log w \cdot n e^{-nw} dw - \left(1 + \frac{\log n}{\gamma}\right) \sigma \int_0^\infty -\log we^{-w} dw$$

$$= \sigma (\gamma + \log n) - \left(1 + \frac{\log n}{\gamma}\right) \sigma \gamma$$

$$= 0$$