Solution for weekly review exercises #1
Chen-Yen Lin
Jan. 12, 2011

(a) 

i. \( \frac{X_1}{X_2} \) or \( \frac{X_1}{|X_2|} \)

ii. \( X_1^2 \) and \( \sum_{i=1}^{5} X_i^2 \)

iii. \( \frac{X_1}{\sqrt{\sum_{i=2}^{4} X_i^2/3}} \)

iv. \( \frac{(X_1^2 + X_2^2)/2}{(X_1^2 + X_2^2 + X_3^2 + X_4^2)/3} \)

v. \( \frac{3}{2} (X_1^2 + X_2^2 + X_3^2 + X_4^2) \)

vi. \( \frac{X_1^2 + X_2^2}{(X_1^2 + X_2^2 + X_3^2 + X_4^2)/2} \)

vii. \( \left[ \frac{X_1^2 + X_2^2}{X_1^2 + X_2^2 + (X_3^2 + X_4^2)} - 0.5 \right] \times 2 \)

(b) 

i. By using (5.4.4), the density function for \( X_{(1)} \) is given by

\[
f_{X_{(1)}}(x) = \frac{n!}{0!(n-1)!} [x^2]^0 [1 - x^2]^{n-1} \frac{2x}{n} = 2nx(1 - x^2)^{n-1}, \quad 0 < x < 1
\]

ii. First realize that \( \frac{1}{\sqrt{2}} \) is the median of the population, then the probability of \( X_{(1)} \) exceeds the median can be computed as follows

\[
P\left( X_{(1)} > \frac{1}{\sqrt{2}} \right) = \int_{\frac{1}{\sqrt{2}}}^{1} 6x(1 - x^2)^2 \, dx = \ldots = \frac{1}{8}
\]

iii. By using (5.4.7), the joint density for \( X_{(2)} \) and \( X_{(3)} \) is given by

\[
f_{X_{(2)}, X_{(3)}}(x_2, x_3) = \frac{3!}{1!0!0!} [x_2^2]^2 2x_2 2x_3 = 24x_2^3 x_3, \quad 0 < x_2 < x_3 < 1
\]

It follows that the expectation of \( X_{(2)} X_{(3)} \) is given by

\[
EX_{(2)} X_{(3)} = \int_0^1 \int_0^{x_3} 24x_2^4 x_3^2 \, dx_2 \, dx_3 = \ldots = \frac{6}{10}
\]

One can also verify the expectations of \( X_{(2)} \) and \( X_{(3)} \) are \( \frac{24}{35} \) and \( \frac{6}{7} \), respectively. Thus, the covariance is \( \frac{3}{5} - \frac{24}{35} \frac{6}{7} \).