Solution for weekly review exercises #10

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Mar. 30, 2011

(a) i. 

\[
\pi(p|x) \propto f(x|p) \pi(p) \propto p^{\alpha-1} (1-p)^{\beta-1} p^n (1-p)^{\sum_{i=1}^n x_i} = p^{\alpha+n} (1-p)^{\sum_{i=1}^n x_i + \beta-1}
\]

Thus, \( p|x \) has Beta\((n+\alpha, \sum_{i=1}^n x_i + \beta)\) distribution.

ii. Using square error loss, the Bayes estimator is the posterior mean. That is

\[
\hat{p}_{Bayes} = \frac{n+\alpha}{n+\sum_{i=1}^n x_i + \alpha + \beta}
\]

iii. As \( n \to \infty \), we can observe that \( \hat{p}_{Bayes} \to \frac{1}{1 + \bar{x}} \)

In addition, one can show that the log-likelihood function is given by

\[
\log L(p|x) = n \log p + \sum_{i=1}^n x_i \log(1-p)
\]

By taking derivative w.r.t \( p \) and set to 0, we can solve the MLE as follow

\[
\frac{d \log L}{dp} = \frac{n}{p} - \frac{\sum_{i=1}^n x_i}{1-p} = 0 \quad \Rightarrow \quad \hat{p}_{MLE} = \frac{1}{1 + \bar{x}}
\]

(b) i. The likelihood function is monotone decreasing function, so the MLE of \( \theta \) is \( \hat{\theta} = X_{(n)} \). Also from text example 6.2.23, the largest order statistic \( X_{(n)} \) is a complete sufficient statistic. Thus, the UMVUE of \( \theta \) is given by \( \frac{n+1}{n} X_{(n)} \)

ii. Derive the CDF of \( n(\theta - \hat{\theta}) \)

\[
P[n(\theta - \hat{\theta}) < c] = P[\hat{\theta} > \theta - \frac{c}{n}]
\]

\[
= 1 - P[X_{(n)} < \theta - \frac{c}{n}]
\]

\[
= 1 - \left[ P(X_1 < \theta - \frac{c}{n}) \right]^n
\]

\[
= 1 - \left[ 1 - \frac{c}{n\theta} \right]^n
\]

\[
\rightarrow 1 - e^{-\frac{c}{\theta}}
\]

Thus, \( n(\theta - \hat{\theta}) \) converges to exponential distribution with mean \( \theta \).
iii. Similar as (ii), we can show that $n(\theta - \hat{\theta})$ converges in distribution
\[ n(\theta - \hat{\theta}) = n(\theta - \hat{\theta} + \hat{\theta} - \hat{\theta}) = n(\theta - \hat{\theta}) + n \left( X_{(n)} - \frac{n+1}{n} X_{(n)} \right) = n(\theta - \hat{\theta}) - X_{(n)} \xrightarrow{d} \exp(\theta) - \theta \]

As a result, the asymptotic MSE for $\hat{\theta}$ and $\tilde{\theta}$ are $E[\exp(\theta)^2] = \frac{2}{n^2}$ and $E[\exp(\theta) - \theta]^2 = \frac{1}{n^2}$, respectively. Therefore, the asymptotic relative efficiency (ARE) of the MLE with respect to the UMVUE is $\frac{1}{2/n^2} = \frac{1}{2}$. That is, UMVUE is asymptotically more efficient than MLE.

(c) First derive the MLE of $\theta$.

\[ L(\theta|x) = \theta^n \left[ \prod_{i=1}^{n} (1 - x_i) \right]^{\theta - 1} \]

\[ \log L = n \log \theta + (\theta - 1) \sum_{i=1}^{n} \log(1 - x_i) \]

\[ \frac{d \log L}{d \theta} = 0 : \frac{n}{\theta} + \sum_{i=1}^{n} \log(1 - x_i) = 0 \Rightarrow \hat{\theta} = -\sum_{i=1}^{n} \log(1 - x_i) \]

The LRT rejects the null hypothesis if and only if

\[ \lambda = \frac{L(\theta_0|x)}{L(\hat{\theta}|x)} < c \]

\[ \Leftrightarrow \frac{\theta_0^n \left[ \prod_{i=1}^{n} (1 - x_i) \right]^{\theta_0 - 1}}{\hat{\theta}^n \left[ \prod_{i=1}^{n} (1 - x_i) \right]^{\hat{\theta} - 1}} > c \]

\[ \Leftrightarrow \hat{\theta}^n \left[ \prod_{i=1}^{n} (1 - x_i) \right]^{\hat{\theta} - 1} > c' \]

\[ \Leftrightarrow n \log \hat{\theta} + (\hat{\theta} - 1) \left[ \sum_{i=1}^{n} \log(1 - x_i) \right] > \log(c') \]

\[ \Leftrightarrow n \log \hat{\theta} - n + \frac{n}{\hat{\theta}} > \log(c_1) \]

\[ \Leftrightarrow \log(\hat{\theta}) + \frac{1}{\hat{\theta}} > c'' \]

\[ \Leftrightarrow \hat{\theta} < c_1 \text{ or } \hat{\theta} > c_2 \]