1. (10 pts) Assume \( \tilde{n} = (n_1, n_2, \ldots, n_c)^T \sim \text{multinomial}(n, \pi) \) \((c > 1)\). We are interested in deriving the score test for testing \( H_0 : \pi = \pi_0 \) v.s. \( H_a : \pi \neq \pi_0 \), where \( \pi_0 = (\pi_1, \pi_2, \ldots, \pi_c)^T \) is a completely known vector such that \( \pi_j > 0 \) and \( \sum_{j=1}^c \pi_j = 1 \). Since \( \sum_{j=1}^c \pi_j = 1 \), there are only \( c - 1 \) many un-constrained parameters. Let us work with \( \pi_1, \pi_2, \ldots, \pi_{c-1} \) and still denote them by \( \pi = (\pi_1, \pi_2, \ldots, \pi_{c-1})^T \). Do the following:

(a) Show the score vector is

\[
U = \frac{\partial \ell(\pi; \tilde{n})}{\partial \pi} = \begin{bmatrix}
n_1/\pi_1 - n_c/\pi_c \\
n_2/\pi_2 - n_c/\pi_c \\
\vdots \\
n_{c-1}/\pi_{c-1} - n_c/\pi_c
\end{bmatrix}.
\]

(b) Show the variance of \( U \):

\[
V = \text{var}(U) = n[\text{diag}\{\pi_1^{-1}, \pi_2^{-1}, \ldots, \pi_{c-1}^{-1}\} + \pi_c^{-1}11^T],
\]

where \( 1 \) is a \((c - 1) \times 1 \) vector of ones.

(c) Show that the score statistic

\[
U^TV^{-1}U|_{H_0} = \sum_{j=1}^c \frac{(n_j - n\pi_{j0})^2}{n\pi_{j0}}
\]

is the Pearson \( \chi^2 \) test statistic for \( H_0 \).

**Hint:** Use the formula \((I + PQ)^{-1} = I - P(I + QP)^{-1}Q\) in finding \( V^{-1} \).

2. (10 pts) Problem 1.9 of CDA (2nd ed.) or Problem 1.10 of CDA (3rd ed.).

**Hint:** Let \( Y = \# \) of deaths in a corps-year. What we are asked to test is \( H_0 : Y \) is from Poisson(\( \mu \)). The original data is \( Y_1, Y_2, \ldots, Y_n \), where \( n = 200 \) corps-years. What is given in Table 1.3 is the number of \( Y_i \)'s that are 0, 1, 2, 3, 4 or \( \geq 5 \). Let \((n_0, n_1, n_2, n_3, n_4)^T\) be the number of \( Y_i \)'s that are 0, 1, 2, 3, or \( \geq 4 \) (truncated at 4). Then \((n_0, n_1, n_2, n_3, n_4)^T\)
has a multinomial distribution. Under $H_0$, the corresponding probabilities are given by the Poisson distribution:

$$\pi_0 = e^{-\mu}, \pi_1 = \mu e^{-\mu}, \pi_2 = \frac{\mu^2 e^{-\mu}}{2!}, \pi_3 = \frac{\mu^3 e^{-\mu}}{3!}, \pi_4 = 1 - (\pi_0 + \pi_1 + \pi_2 + \pi_3).$$

We can then use the Pearson $\chi^2$ test to test this null hypothesis. In order to construct the test, we need to find out the MLE of $\mu$ under $H_0$. Try to find it out from the original un-truncated data.

3. (10 pts) Problem 1.30 of CDA (2nd ed.) or Problem 1.29 of CDA (3rd ed.).

4. (7 pts) Problem 2.1 of CDA (2nd ed.) or Problem 2.3 of CDA (3rd ed.).

5. (6 pts) Problem 2.8 of CDA (2nd ed.) or Problem 2.10 of CDA (3rd ed.).

6. (7 pts) Problem 2.10 of CDA (2nd ed.) or Problem 2.11 of CDA (3rd ed.).