Practice Exercises for Final Exam
ST 521 - Statistical Theory - I

The ACTUAL exam will consists of <u>less</u> number of problems.

1. Suppose $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f_X(x)$, where $f_X(x)$ is the pdf of a random variable X.

   (a) Show that $U = F_X(X) \sim U(0,1)$, where $F_X(x) = \int_{-\infty}^{x} f_X(t)dt$.

   (b) Define $U_i = F_X(X_i)$ for $i = 1, 2, \ldots, n$. Show that $U_1, U_2, \ldots, U_n \overset{iid}{\sim} U(0,1)$.

   (c) Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics corresponding to the random sample of size n from $f_X(x)$. Define $U_{(i)} = F_X(X_{(i)})$ for $i = 1, 2, \ldots, n$. Show that $U_{(1)}, \ldots, U_{n)}$ are the order statistics corresponding to $U_1, \ldots, U_n$.

   (d) Show that $F_{U_{(n)}}(u) = \Pr[U_{(n)} \leq u] = u^n$ if $0 < u < 1$ and hence show that $U_{(n)} \sim Beta(n, 1)$. Obtain the mean and variance of $U_{(n)}$.

   (e) Use (c) and (d) to obtain the cdf and pdf of $X_{(n)}$.
   Show that $E[X_{(n)}] = n \int_0^1 F_X^{-1}(u) u^{n-1} du$, where $F_X^{-1}(u) = \inf\{x : F_X(x) \geq u\}$

2. Suppose $X_i \overset{indep}{\sim} N(i-3, i^2)$ for $i = 1, \ldots, 5$. Use X_i's to construct a statistic with the indicated distribution.

   (a) a $N(0,1)$ distribution,

   (b) a χ^2_5 distribution,

   (c) a t_5 distribution,

   (d) a $F_{1,5}$ distribution,

   (e) a $Gamma(2,3)$ distribution,

   (f) a $U(0,1)$ distribution,

   (g) a $Beta(1,2)$ distribution,

   (h) a $DE(0,1)$ distribution,

   (i) a $Geometric(0.75)$ distribution,

   (j) a $NB(5, 0.25)$ distribution, and

   (k) a $F_{3,2}$ distribution.

3. Let (X, Y) be a pair of ransom variables with finite variances.

   (a) Show that $\hat{g}(x) = E[Y|X = x]$ minimizes $E[Y - g(X)]^2$ over all functions $g(x)$ such that $E[g^2(X)] < \infty$.

   (b) Let $\epsilon = Y - \hat{g}(X)$. Show that

      i. $E[\epsilon] = 0$ and $Var[\epsilon] = E[Var[Y|X]]$;

      ii. $Cov[X, \epsilon] = 0$; and more generally,

      iii. ϵ is uncorrelated with any function $g(X)$ such that $E[g^2(X)] < \infty$.

   (c) Show that $Var[Y] = Var[\epsilon] + Var[\hat{g}(X)]$.

   (*in above* $\hat{g}(x)$ *is known as the least-squares regression function*)

4. Suppose X and Y are independent random variables with finite variances.

   (a) Show that $Corr[X, Y] = 0$.

   (b) Show that $Corr[|X|, \sqrt{|Y|}] = 0$.

   (c) Show that $Corr[2X - 3Y, 3X - 2Y] > 0$.

   (d) Show that $Corr[2X + 3Y, 3X - 2Y] = 0$ if $Var[X] = Var[Y]$.

   (e) Show that $Corr[g(X), h(Y)] = 0$, provided $g(X)$ and $h(Y)$ have finite variances.

5. Suppose X and Y are uncorrelated random variables with finite variances.

   (a) Are X and Y necessarily independent?

   (b) If the joint distribution of (X, Y) is a bivariate normal distribution, show that X and Y are independent.

   (c) Given an example of a pair of uncorrelated random variables that are not independent.

   (d) Suppose $U_1 = a_1 + a_2X$ and $U_2 = b_1 + b_2Y$. Show that $Corr(U_1, U_2) = 0$. Find a_1, a_2, b_1, b_2 (in terms of means and variances of X and Y) such that $E[U_1] = E[U_2] = 0$ and $Var[U_1] = Var[U_2] = 1$.

   (e) Show that $c_1U_1 + c_2U_2$ is positively correlated with U_1 iff $c_1 > 0$.

6. Suppose that $X_1, X_2 \mid Z = z \overset{iid}{\sim} f(x|z)$, i.e., X_1 and X_2 are *conditionally independent and identically distributed (ciid)* random variables given the random variable Z.
   (a) Find the (unconditional) joint density of (X_1, X_2).
   (b) Show that X_1 and X_2 are identically distributed.
   (c) Assume that $Var[X_1]$ is finite. Compute $Corr[X_1, X_2]$ and conclude that X_1 and X_2 are *always* positively correlated.
   (d) Can X_1 and X_2 be independent?
   (e) Can X_1 and X_2 be perfectly correlated? (i.e. $Corr[X_1, X_2] = 1$?)
7. Suppose $U_1, \ldots, U_n \overset{iid}{\sim} U(0, 1)$.
   (a) Show that $-2\log(U_1) \sim \chi^2_2$.
   (b) Obtain the pdf of the geometric mean $(\prod_{i=1}^n U_i)^{1/n}$.
   (c) Let $U_{(n)} = \max_{1\le i\le n} U_i$ and $U_{(1)} = \min_{1\le i\le n} U_i$ be the maximum and minimum order statistics. Obtain the joint pdf of $(U_{(1)}, U_{(n)})$.
   (d) Compute $Corr[U_{(1)}, U_{(n)}]$.
   (e) Obtain the pdf of the range, $R = U_{(n)} - U_{(1)}$.
   (f) Compute $E[R]$ and $Var[R]$.
8. Let X be a positive valued random variable with finite variance. Show that
   (a) $Var[\log(X)] < \infty$ and $Var[\sqrt{X}] < \infty$.
   (b) $Corr[\log(X), X] \ge 0$.
   (c) $Corr[\frac{1}{1+X}, \sqrt{X}] \le 0$.
   (d) $\left(E[\frac{1}{X}]\right)^{-1} \le e^{E[\log(X)]} \le E[X]$. When does the equality holds?
   (*this is the famous inequality H.M.* $\le$ *G.M.* $\le$ *A.M.*)
9. Give an example of a pdf
   (a) that belongs to both exponential family and location and scale family,
   (b) that belongs to an exponential family but not to a location and scale family,
   (c) that belongs to a location and scale family but not to an exponential family.
10. Review **all** the problems solved during Lab hours (ST 521L) and in PP-I.