Practice Final Exam, Fall 2006, Statistics 521

1. Let X have the log-normal distribution having the density

$$f(x) = \frac{1}{\sqrt{2\pi}x} e^{-(\log x)^2/2}, \quad 0 < x < \infty.$$

Find the density function of $Y = \sqrt{X}$.

2. Let X have the standard Cauchy density function

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Find the density function of $Y = 1/X$ and the median of $Z = Y^2 = 1/X^2$.

3. Let (X, Y) have joint density $f(x, y) = x^{-2}y^{-2}I\{x > 1, y > 1\}$. Compute the joint density of $U = XY$ and $V = X/Y$. Also obtain the marginal density of U. [You may find it useful that $U^{-1} < V < U$.]

4. Let the joint density of (X, Y) be

$$f(x, y) = \frac{1}{8}(x^2 - y^2)e^{-x}, \qquad 0 < x < \infty, \quad -x < y < x.$$

Find the covariance of X and Y.

5. Let X_1, X_2, X_3, X_4 be independent $N(0, 1)$. Show that

$$T = \frac{\sqrt{3}(X_1 + X_2 + X_3 + X_4)}{\sqrt{(X_1 - X_2 + X_3 - X_4)^2 + (X_1 + X_2 - X_3 - X_4)^2 + (X_1 - X_2 - X_3 + X_4)^2}}$$

has t-distribution with 3 degrees of freedom. Calculate the variance of T.

6. Let X_1, X_2, X_3, X_4 be i.i.d. $N(0, 1)$. Show that

$$\frac{(X_1 + X_2 + X_3 + X_4)^2 + (X_1 - X_2 + X_3 - X_4)^2)}{(X_1 + X_2 - X_3 - X_4)^2 + (X_1 - X_2 - X_3 + X_4)^2}$$

has F distribution with $(2, 2)$ degrees of freedom.

7. Let $X_1, \cdots, X_n$ be n independent observations from the density $f(x) = \frac{1}{2}x^2e^{-x}$ and let $T_n = \frac{n}{\sum_{i=1}^n X_i^{-1}}$. Find constants a and b such that $\sqrt{n}(T_n - a)/b$ converges in distribution to $N(0, 1)$ as $n \to \infty$. [Hint: Consider $Y_i = X_i^{-1}$ and use the delta method.]

8. Let $X_1, \ldots, X_n$ be Normal$(0, \theta)$ and $Y_1, \ldots, Y_n$ be Normal$(0, 1)$, and let all the variables be mutually independent. Consider

$$V_n = \frac{X_1^2 + \cdots + X_n^2}{Y_1^2 + \cdots + Y_n^2}.$$

(a) Show that

$$E(V_n) = \frac{n\theta}{n-2}.$$

In this derivation, the fact that

$$\int_0^\infty x^{a-1} e^{-x/b} dx = b^a \Gamma(a), \qquad \text{for all } a > 0,\ b > 0,$$

may be useful.

(b) Using the fact that

$$V_n - \theta = \frac{\sum_{i=1}^n (X_i^2 - \theta Y_i^2)}{\sum_{i=1}^n Y_i^2},$$

show that $\sqrt{n}(V_n - \theta)$ converges in distribution to Normal $(0, 4\theta^2)$.

(c) Is it true that $\sqrt{n}(V_n - E(V_n))$ converges in distribution to Normal$(0, 4\theta^2)$? Justify your answer.

(d) Obtain the asymptotic distribution of $\log V_n$.

9. A shooter hits a target with probability p independently in each attempt. She decides to hit the target r times. Let X stand for the (random) number of attempts she needs. From the first principle, derive the probability mass function of X.

Find the moment generating function $m(t) = E(e^{tX})$ of X. For what values of t will $m(t)$ be finite?

If $r \to \infty$ and $p \to 1$ such that $r(1-p) \to \lambda$, where $0 < \lambda < \infty$, show that the distribution of $X - r$ can be approximated by the Poisson distribution with parameter λ. [Hint: Calculate the moment generating function of $X - r$ and identify its limit.]

If $r \to \infty$ and $0 < p < 1$ remains fixed, show that the distribution of $r^{-1/2}(X - r/p)$ can be approximated by a normal distribution. What are the mean and variance of the approximating normal distribution? Clearly mention any theorem you are using.