A Simple Linear Boolean Model for Mass Flow

Jason A. Osborne, North Carolina State University
Tony E. Grift, University of Illinois
Jason A. Osborne, Box 8203, Department of Statistics, NCSU, Raleigh, NC 27695

Key Words: Coverage processes, linear boolean models, queueing.

1. Introduction

Assessing the flow of granular fertilizer or pesticide from an aerial spreader duct presents a challenging mass flow problem. The lack of a direct feedback mechanism for granular spreaders makes uniform distribution over a field difficult, as there are many uncontrollable factors, such as wind speed, that cause variability in flow rates. Existing technologies for counting the passage of individual particles, or clumps of multiple particles, past a sensor include those based on a single light source with an optically matched receiver and a system based on a spatially non-coherent light source (Grift et al. (2001).) These systems involve a sensor which goes on and off as with passing clumps of particles. The sensor turns on when one end of a particle or clump of particles passes a detection point and does not go off until the particle or clump of particles has passed the detection point completely. If each particle takes the same known amount of time to pass the detection point, the system forms what is called a type II counter (Pyke (1958).) The observed data from such a counter are the time periods that the sensor is on, called clumplengths or busy periods, and the time periods that the sensor is off, called spacings. Accurate assessment of particle flow from data such as these could facilitate uniform distribution of granular pesticide or fertilizer across a targeted field.

A simple linear boolean model described by Hall (1988) is proposed for the sequence of clumplengths. Methods of statistical inference are developed. Interest centers on prediction of unobservable random particle flow, conditional on observed clumplengths as well as on estimation of the rate at which particles are flowing. Experiments conducted by Grift et al. (2001) involving known numbers of BBs of approximately equal shape and diameter are described. The data are analyzed and used to evaluate the performance of the methodology.

2. The boolean model

To obtain a probability model for the mass flow process, assume particles arrive at the sensor at times $W_1 < W_2 < W_3 \cdots < W_{A(t)}$ according to a homogeneous Poisson process with unknown rate λ. Here $A(t)$ denotes the total number of particles which arrive at the system by time t. It is the quantity whose prediction is of primary interest. Assume particles have known, constant diameter and are moving at a common known velocity and passage continues unabated upon arrival at the sensor. Under these assumptions, the time, say a, each particle takes to pass through the sensor is constant and known. and the system forms an $M/D/\infty$ queue with constant service times.

Let $Y_1, Y_2, \ldots$ denote the sequence of clumplengths and $Z_1, Z_2, \ldots$ the spacings between them. Let $N(t)$ denote the number of complete clumps observed by time t. Suppose, for example, that $N(t) = 6$ particles of identical diameter $a = 4.5$ arrive in the first $t = 30$ time units at times $(w_1, \ldots, w_6) = (0, 0.5, 10.1, 13.0, 13.8, 24.6)$. This process would yield three complete clumps, of lengths $(y_1, y_2, y_3) = (5.0, 8.2, 4.5)$ and spacings $(z_1, z_2, z_3) = (5.1, 6.4, 0.9)$. A diagram of the clumplengths that would be observed with such a process appears in Figure 1. Arrivals are denoted by $\times$. Interest is in much longer sequences than the one shown in the figure, but it illustrates the observation process and notation.

3. Prediction of $A(t)$

Consider the case where either λ is known or variance in its estimation is negligible. Mean particle flow is $E[A(t)] = \lambda t$. A naive prediction is $\hat{A}_0(t; \lambda) = \lambda t$. When the spacings and t are missing, another naive prediction can be formed by substitution of $t \approx \sum Y_i + \sum E(Z_i)$ into the expression giving $\hat{A}_0(\tilde{t}) = \lambda \sum Y_i + N(t)$.

Let the clump order K_i be the unobservable number of particles comprising clump i. A clump is a singleton $K_i = 1$ if there are no arrivals within a time units of the start of the clump, which occurs with probability $e^{-\lambda a}$. A clump is a doubleton if there is exactly 1 arrival within a units and none

Figure 1: Simple linear Boolean process

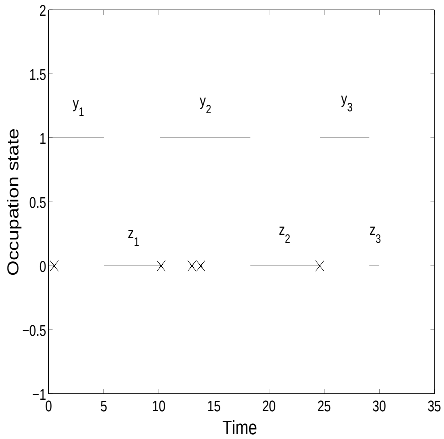
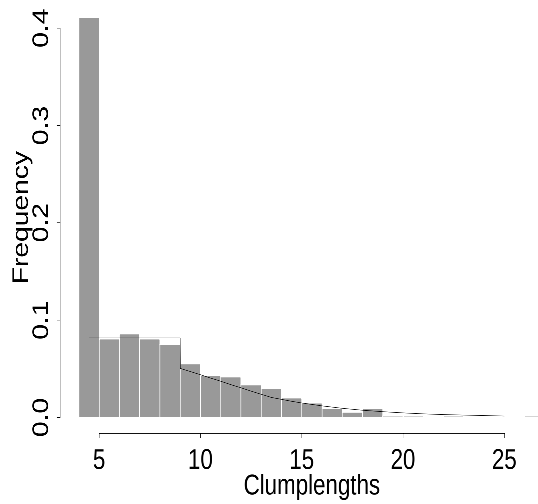


Figure 2: Histogram of clumplengths from 2000 BBs, $\alpha = 4.5$



thereafter, an event which occurs with probability $(1 - e^{-\lambda a})e^{-\lambda a}$ and so on. Hence, K_i has a geometric distribution with support on positive integers:

$$\Pr(K_i = k) = (1 - e^{-\lambda a})^{k-1} e^{-\lambda a} \quad \text{for } k = 1, 2, \ldots$$

with $E(K_i) = e^{\lambda a}$ and $\text{Var}(K_i) = e^{2\lambda a} - e^{\lambda a}$. Total particle flow can be expressed as a sum of a random number of i.i.d. clump orders, or a compound sum:

$$A(t) = \sum_{i=1}^{N(t)} K_i.$$

Wald's equation (Ross (1996), Ch. 3) yields $E[A(t)] = E[N(t)]e^{\lambda a}$ which leads to the predictor $\hat{A}_1(t; \lambda) = N(t)e^{\lambda a}$.

Total particle flow can also be written

$$A(t) = \sum_{i:K_i=1} 1 + \sum_{i:K_i>1} K_i.$$

Let M_1 denote the number of singletons. Another predictor, $\hat{A}_2(t; \lambda)$ based on this observable statistic can be constructed from the expression above as

$$\begin{aligned} \hat{A}_2(t; \lambda) &= M_1 + (N(t) - M_1)E(K|K > 1) \\ &= M_1 + (N(t) - M_1)(1 + e^{\lambda a}). \end{aligned}$$

The last equality follows from the memoryless property of the geometric distribution. Grift (2001) studies a predictor $\hat{A}_G(t) = \frac{N(t)^2}{M_1}$. which simultaneously accounts for estimation of $e^{\lambda a}$ using $\frac{N(t)}{M_1}$

More efficient predictors can be constructed by conditioning more explicitly on clumplengths:

$$\hat{A}_3(t) = \sum_1^{N(t)} E(K_i|Y_i).$$

Of course $E[K_i|Y_i = a] = 1$. Suppose that a clump has $y > a$ so that it involves multiple particles. If $[\cdot]$ denotes the largest integer not exceeding the argument, Hall (1988) shows that

$$f(y) = \frac{\lambda e^{-a\lambda}}{1 - e^{-a\lambda}} \left[1 + \sum_{j=1}^{[y/a]} \frac{(-1)^j}{j!} \{\lambda y_j\}^{j-1} e^{-ja\lambda} \{\lambda(y_j) + j\} \right].$$

where $y_j = y - (j+1)a$. To see the shape of the continuous component of the clumplength density, see Figure 2, which is produced from the experimental data described in section 5.

To derive the conditional probability mass function $\Pr(K = k|Y = y)$ for a multi-particle clump, let the beginning of the clump be the origin. The joint event $K = k$ and $Y \in (y, y + dy)$ occurs if and only if there is a particle arrival at $(y - a, y - a + dy)$, no arrival in $(y - a, y)$, exactly $k-2$ arrivals in $(0, y-a)$, and the nearest neighbor of each of these $k - 2$ arrival times is less than a time units away. Since the first three of these conditions are independent and the fourth is conditionally independent of the first two given the third, the joint probability of these four events is the product

$$\lambda dy e^{-\lambda a} \frac{(\lambda(y - a))^{k-2}}{(k - 2)!} e^{-\lambda(y-a)} p_{k-2} \left(\frac{a}{y - a} \right)$$

where $p_n(u)$ denotes the chance that the largest division formed by a random sample of n points taken from the unit interval does not exceed u. This probability can be written

$$p_n(u) = \sum_{j=0}^{[u^{-1}]} (-1)^j \left(\begin{array}{c} n+1 \\ j \end{array} \right) (1-ju)^n$$

Division by $f(y)$ and differentiation with respect to y yields the conditional density

$$\begin{aligned} &\Pr(K=k|Y=y) \\ &= \frac{\lambda e^{-\lambda y}}{f(y)} \frac{(\lambda(y-a))^{k-2}}{(k-2)!} p_{k-2}\left(\frac{a}{y-a}\right) \end{aligned}$$

Summation over k yields the conditional mean:

$$\begin{aligned} &E(K|Y=y) \\ &= \frac{\lambda e^{-\lambda a}}{f(y)} \sum_{k=[y/a]+1}^{\infty} k \frac{(\lambda(y-a))^{k-2}}{(k-2)!} p_{k-2}(\frac{a}{y-a}) \end{aligned}$$

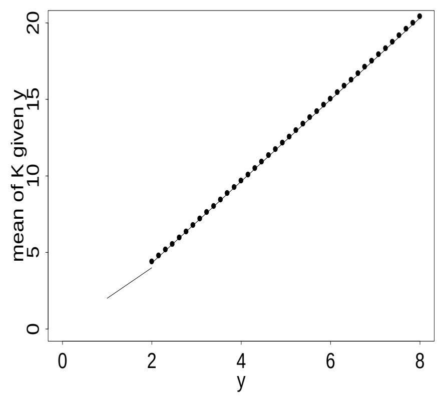
The prediction of an individual clump order which is a function of the clumplength y and minimizes the mean squared error $E[(K-y)^2]$, is the *Bayes* prediction, or $\hat{K}(y) = E(K|y)$. A prediction of $A(t) = \sum K_i$ is then given by $\hat{A}_B(t) = \sum \hat{K}(y_i)$. Inspection of $\Pr(K=k|Y=y)$ reveals that for $a < y < 2a$, K has the translated Poisson distribution with mean $2+\lambda(y-a)$ and variance $\lambda(y-a)$. For larger y, numerical evaluation of this conditional mean can be computationally expensive, as it is an infinite sum of an alternating sequence of products of large and small numbers. To recompute this mean in applications involving thousands of clumps could be prohibitively slow.

Plots of $E(K|Y=y)$ versus y for various values of λ and y suggest a linear association between the conditional mean and y. A scatterplot of $E(K|Y=y)$ clumplength y appears in Figure 3 for the case where $\lambda=2, a=1, E(Y)=3.2$ and $\mathrm{Var}(Y)=6.0$. A line connecting the exact means at $y=2$ and $y=8$ is overlaid on the plot. On $(a,2a]$ the mean is linear in y and there is a jump discontinuity of $\lambda a e^{-\lambda a}(1-e^{-\lambda a})^{-1}$ at $y=2a$. For the value $\lambda a = 2$ the mean is very nearly linear for $y > 2a$. Consider approximating $E(K|Y=y)$ with a linear function of y with

$$\widehat{\mathrm{slope}}(\lambda,a) = \frac{E(K|Y=3a)-E(K|Y=2a)}{a}.$$

Dividing the difference between the mean for $y \in (2a, 2a+\epsilon)$ and the mean for $y \in (3a, 3a-\epsilon)$ by a

Figure 3: Exact conditional mean and linear interpolation, $\lambda=2, a=1$



yields

$$\begin{aligned} &\widehat{\mathrm{slope}}(\lambda,a) \\ &= \lambda + \frac{\lambda e^{-\lambda a}(e^{-\lambda a}-(1-\lambda a)}{(1-e^{-\lambda a}(\lambda a+1)))(1-e^{-\lambda a})}. \end{aligned}$$

Note that for large λa, $\widehat{slope} \approx \lambda$. The slope can be viewed as the arrival rate of particles, λ, plus an additional amount due to the information that there was clumping (since $y > a$).

A predictor $\hat{A}_3(t)$ can be constructing by exploiting the apparent linearity in y and the approximated slope. If the approximate linearity holds, then take

$$\begin{aligned} &E[A(t)|Y_1,\ldots,Y_{N(t)}] \\ &= \sum_{i=1}^{N(t)} E[K_i|Y_i] \\ &= M_1 + \sum_{i:a<Y_i\le 2a} (2+\lambda(Y_i-a)) \\ &+ \sum_{i:Y_i>2a} E[K_i|Y_i] \\ &\approx M_1 + \sum_{i:a<Y_i\le 2a} (2+\lambda(Y_i-a)) \\ &+ \sum_{i:Y_i>a}^{N(t)} \left(2+\frac{\lambda a}{1-e^{-\lambda a}} + \widehat{slope}(\lambda,a)(Y_i-2a)\right) \\ &\equiv \hat{A}_3(t) \end{aligned}$$

Except for $\hat{A}_3(t)$, each of the predictors is approximately unbiased for $A(t)$ for known λ, with the bias being due to the possibility of an incomplete clump at the end of the sequence. Let $A_{N(t)}$ denote particle flow among the $N(t)$ complete clumps. Unbiasedness can be seen by taking expectations of additive prediction errors:

$$\begin{aligned}
\hat{A}_n(t;\lambda) - A(t) &= \lambda t - A(t) \\
\hat{A}_0(\tilde{t}) - A_{N(t)} &= \sum_{i=1}^{N(t)} (\lambda Y_i - 1 - K_i) \\
\hat{A}_1(t;\lambda) - A_{N(t)} &= \sum_{i=1}^{N(t)} (e^{\lambda a} - K_i) \\
\hat{A}_2(t;\lambda) - A_{N(t)} &= \sum_{i:Y_i>a} (1 + e^{\lambda a} - K_i) \\
\hat{A}_G(t) - A_{N(t)} &= \sum_{i=1}^{N(t)} (\frac{N(t)}{M_1} - K_i)
\end{aligned}$$

For finite t, the first four prediction errors above have mean 0 exactly, and the error for $\hat{A}_G(t)$ has approximate mean 0. Since particle flow is approximately normally distributed, prediction intervals can be developed by consideration of the second moment of these prediction errors.

4. Estimation of λ

Approximate likelihood functions can be specified by ignoring the residual lifetime of the process. The residual lifetime is the duration of the last incomplete clump or spacing. By independence of clumplengths, the approximate partial boolean likelihood can be factored as

$$\tilde{\mathcal{L}}(\lambda; y_1, \ldots, y_{N(t)}) = \underbrace{e^{-M_1 a\lambda}}_{\text{singletons}} \underbrace{(1 - e^{-a\lambda})^{N-M_1} \prod_{i:y_i>a} f(y_i;\lambda)}_{\text{complete multi-particle lengths.}}$$

If spacings $z_1, z_2, \ldots$ are available, an approximate complete boolean likelihood is given by

$$\begin{aligned}
&\tilde{\mathcal{L}}(\lambda; y_1, \ldots, y_{N(t)}, z_1, z_2, \ldots) = \\
&\quad \lambda^N e^{-\lambda \sum z_i} e^{-M_1 a\lambda} \\
&\quad \times \quad (1 - e^{-a\lambda})^{N-M_1} \prod_{i:y_i>a} f(y_i;\lambda).
\end{aligned}$$

For large t, the maximum likelihood estimators from these functions are approximately normally distributed with asymptotic variance of the form

$$\{E(\partial^2 \log \tilde{\mathcal{L}} / \partial \lambda^2)\}^{-1}.$$

As with the exact conditional mean $\hat{A}_B(t)$, the observed information can be computationally expensive to evaluate when thousands of clumplengths are observed.

Method-of-moments (MOM) estimators can be constructed using the statistics $N(t)$ and M_1. The i.i.d. sums $(Z_i + Y_i)$ form a renewal process. Elementary renewal theory (Ross (1996), ch. 3) yields that for large t,

$$\frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} \xrightarrow{\mathcal{L}} N(0,1)$$

where $\mu = E(Z+Y) = \lambda^{-1} e^{\lambda a}$ and $\sigma^2 = \mathrm{Var}(Z+Y) = \lambda^{-2}(e^{2\lambda a} - 2\lambda a e^{\lambda a})$ are moments of a randomly sampled renewal period. Moments for $N(t)$ are then

$$\begin{aligned}
E[N(t)] &\approx \lambda t e^{-\lambda a} \\
\mathrm{Var}[N(t)] &\approx \lambda t \left(e^{-\lambda a} - 2a\lambda e^{-2a\lambda}\right)
\end{aligned}$$

The probability that a randomly selected clump is a singleton is $e^{-\lambda a}$ so that $E(M_1) = \lambda a e^{-2\lambda a}$. A MOM estimator constructed from these moments is

$$\tilde{\lambda}_1 = -\frac{1}{a} \log\left(\frac{M_1}{N(t)}\right).$$

It can be shown that as $t \to \infty$, the variance of $M_1/N(t)$ is $(1 - e^{-\lambda a})(\lambda t)^{-1} + O(t^{-1})$. Taking conditional expectation given $N(t)$ of a series expansion of $-a^{-1}\log[M_1/N(t)]$ yields a conditional standard error which can be estimated by

$$SE[\tilde{\lambda}_1 | N(t)] \approx \sqrt{\frac{e^{\tilde{\lambda}_1 a} - 1}{a^2 N(t)}}.$$

In the situation of small λa or light clumping, $\tilde{\lambda}_1$ can be improved upon using the information contained in $N(t)$ through the relation $E[N(t)] \approx \lambda t e^{-\lambda a}$ which leads to the estimator

$$\tilde{\lambda}_2 = \frac{N(t)}{t} e^{\tilde{\lambda}_1 a}.$$

For fixed $N(t)$, $\tilde{\lambda}_2$ is a differentiable function of $\tilde{\lambda}_1$ and so has asymptotic conditional variance

$$\mathrm{Var}[\tilde{\lambda}_2 | N(t)] \approx \left(\frac{N(t)}{t}\right)^2 a^2 e^{2\lambda a} \mathrm{Var}(\tilde{\lambda}_1).$$

Using moments of $N(t)$, the relative efficiency of $\tilde{\lambda}_1$ to $\tilde{\lambda}_2$ can be shown to depend upon λa:

$$\begin{aligned}
&E\left[a^2 e^{2\lambda a} \left(\frac{N(t)}{t}\right)^2\right] \\
&\quad = (a/t)^2 e^{2\lambda a} (\mathrm{Var}[N(t)] + E[N(t)]^2) \\
&\quad = (\lambda a)^2 + O(t^{-1})
\end{aligned}$$

Other estimators of λ can be constructed by consideration of *vacancy* or total time that the sensor is unoccupied. If vacancy V is defined by

$$V = \sum_{1}^{N(t)} Z_i + R$$

where R denotes the remainder of time between the end of the last complete clump and the minimum of t and the starting point of a possible final incomplete clump. It is easily shown that

$$E(V) = \int_0^t \Pr(\text{sensor on at time } x)dx = te^{-\lambda a}.$$

A vacancy-based MOM estimator is then $\tilde{\lambda}_{V_1} = \alpha^{-1}\log\frac{t}{V}$. An estimator can also be derived from the spacings, $Z_1, Z_2, \ldots$ which are i.i.d. exponential with mean λ^{-1}, so that $\tilde{\lambda}_{V_2} = \bar{Z}^{-1}$. Hall (1988) develops asymptotic theory for these statistics.

An important issue in estimation of λ is robustness to departures from the model. Inspection of experimental data collected by Grift (2002) reveals that the number of clumplengths slightly above and below the known mean particle diameter of $\alpha = 4.5mm$ is slightly more than expected. This *leakage* can be caused by variable particle diameters or velocities or errors with the measurement device. One approach to handling clumps with lengths less than a, which cannot occur under the simple linear boolean model, is to classify them as singletons. When singletons with lengths in excess of a occur, there is no way to distinguish them from multi-particle clumps, so they are misclassified. If the assumption that equal particles diameters is relaxed and it is only assumed that the diameters are i.i.d with mean α, then

$$E(Y;\lambda) = \frac{e^{\lambda\alpha} - 1}{\lambda}.$$

The MOM estimator obtained by solving $E(Y;\lambda) = \bar{Y}$ is robust to nonconstant diameters. A solution exists by the mean value theorem with $E(Y;\lambda)$ increasing in λ. Though there is no analytic solution, the equation can be solved rapidly using statistical software, such as the `uniroot` function in $S+$.

Simulations with parameter values similar to those in the experiments provide some information about performance. Empirical coverages for confidence intervals based on normal approximations with estimated standard errors for the five MOM estimators are close to nominal. Intervals from $\tilde{\lambda}_3$ using bootstrap standard errors have good coverage. Normal plots and Kolmogorov-Smirnov statistics do not indicate any obvious non-normality for studentized versions of any of the estimates with the exception of $\tilde{\lambda}_2$, which is highly skewed in some cases. In applications where the vacancy-based estimator $\tilde{\lambda}_{V_1}$ can be computed, it appears to outperform all other moment-based estimators considered here, but not dramatically.

Table 1: Clumplength frequencies

	Clumplength						
Rep.	≤ 4	4.5	5	5.5	6	6.5	≥ 7
1	6	259	52	29	38	38	324
2	10	281	60	31	38	35	337
3	12	258	45	37	34	28	363
4	12	266	54	38	36	41	327
5	7	257	42	33	27	38	341

Table 2: Statistics and estimates from $A(t) = 2000$

Rep.	$N(t)$	M	$\bar{Y}$	$\tilde{\lambda}_3$	$\tilde{\lambda}_{V_1}$
1	746	265	7.55	0.21	0.23
2	792	291	7.25	0.20	0.22
3	777	270	7.47	0.21	0.23
4	774	278	7.31	0.20	0.23
5	745	264	7.58	0.21	0.23

5. Experimental data

Table 1 summarizes the clumplengths from an experiment (Grift (2002)) in which 2000 BBs, each approximately $4.5mm$ in diameter are dropped through a device simulating an aerial spreader duct. The experiment is replicated 5 times. Statistics and estimates of λ appear in Table 2. The partial boolean MLE (not shown) and the root MOM estimator agree to the nearest 0.01. A probability histogram of the clumplengths from one replication of the experiment is shown in Figure 2. The estimated density of the continuous component of clumplengths, using $\tilde{\lambda}_3 = 0.21$ is also plotted on this figure. The empirical clumplength distributions from the other four replications of the experiment look the same.

Though the histograms do not indicate any lack of fit of the simple linear boolean model, inspection of Table 1 does reveal a small problem with *leakage*. If the model holds, clumplengths should be uniformly distributed on $(a, 2a) = (4.5, 9)$. There are moderately larger counts for the cell with midpoint 5 than expected in the table. The assumption that particle diameters do not vary about their mean of $4.5mm$

Table 3: Predictions of $A(t) = 2000$, based on $\tilde{\lambda}_3$

Rep.	$\hat{A}_G(t)$	$\tilde{A}_0(t)$	$\hat{A}_2(t)$	$\hat{A}_3(t)$
1	1905.9	1944.8	1999.9	1971.2
2	1960.2	1924.2	2009.2	1968.4
3	2053.5	1990.8	2076.0	2029.0
4	1977.1	1907.1	1996.1	1945.4
5	1920.5	1957.7	2009.0	1987.8
Mean	1963.4	1944.9	2018.1	1980.4
Var	3366.6	1030.8	1082.7	968.6
RMSE	63.5	62.1	34.5	34.1

is not realistic. When singletons are longer than $4.5mm$, rounded to the nearest $0.5mm$, a positive bias results in prediction of $A(t)$ and estimation of λ because they are viewed as multi-particle clumps. In these analyses, clumplengths that were observed to be less than $4.5mm$ after rounding, shown in the leftmost column of Table 1, were classified as singletons and contributed to M_1.

One approach to handling this problem is to classify clumps as singletons after bumping the deterministic diameter up by some amount, such as by one or two standard deviations. Grift (2002) considers predictions using $a \approx 4.7$ and gets good performance. Computations with several values of $a > 4.5$ for these data reveal a strong sensitivity of the predictors to "choice" of a. The robust root estimator $\tilde{\lambda}_3$ needs no such adjustment, as it is unbiased even after relaxing the assumption of constant diameters, and requires only their mean. Some predictions based on $\tilde{\lambda}_3$ along with $A_G(t)$ using $a = 4.7$ are shown in Table 3 along with root mean squared errors (RMSE). The predictions $\hat{A}_0(t; \tilde{\lambda}_3)$ and $\hat{A}_0(\tilde{t}, \tilde{\lambda}_3)$ agree to the nearest 0.1, so only the former is displayed.

Performance summaries of predictors computed from data from another experiment involving 4000 particles dropped into the simulator from four different heights appear in Table 4. There are 10 replications of the experiment for each height. A plot of the numbers of particles versus replication appears in Figure 4. Different heights result in different flow rates and simulate different conditions for aerial spreaders. The average root estimate over the 10 replications at each of the four heights is $\tilde{\lambda}_3 = 0.06, 0.08, 0.12, 0.16$, respectively.

For the experiments using 2000 and 4000 particles, the linear approximation to the conditional mean particle flow based on particle clumplengths appears to do well in predicting $A(t)$. The root estimator $\tilde{\lambda}_3$, which is inexpensive to compute, also performs well here and enjoys robustness to conditions where either particles diameters have substantial variability or are moving past the mass flow sensor at substantially different velocities.

Figure 4: Clump counts with $A(t) = 4000$

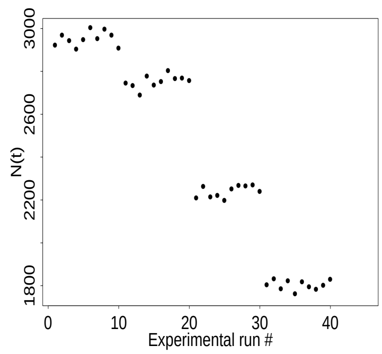


Table 4: RMSE for $A(t) = 4000$ expts

	RMSE	RMSE using $\tilde{\lambda}_3$		
Reps	$\hat{A}_G(t)$	$\tilde{A}_n(t)$	$\hat{A}_2(t)$	$\hat{A}_3(t)$
1-10	28.1	128.4	50.0	40.5
11-20	24.3	222.6	94.6	19.9
21-30	81.9	159.1	103.6	30.5
31-40	64.4	84.0	44.8	48.7

References

Grift, T. (2001), "Mass Flow Measurement of Granular Materials in Aerial Application - Part 1: Simulation and Modeling," *Transactions of the American Society of Agricultural Engineers*, 44, 19–26.

— (2002), "Cluster Criteria," *Resource Engineering*, 9, 9–10.

Grift, T., Walker, J., and Hofstee, J. (2001), "Mass Flow Measurement of Granular Materials in Aerial Application - Part 2: Experimental Model Validation," *Transactions of the American Society of Agricultural Engineers*, 44, 27–34.

Hall, P. G. (1988), *Introduction to the Theory of Coverage Processes*, U.S.A.: John Wiley & Sons, Inc.

Pyke, R. (1958), "On Renewal Processes Related to Type-I and Type-II Counter Models," *The Annals of Mathematical Statistics*, 29, 737–754.

Ross, S. (1996), *Stochastic Processes, 2nd edition*, USA: John Wiley & Sons.