Prob 5. Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with an exponential distribution

$$f(x|\theta) = \theta e^{-\theta x}, \quad \text{where} \quad x > 0, \theta > 0.$$

Suppose the prior distribution of θ is Gamma(α, β).

(a) Find the posterior density function of θ given $X_1, ..., X_n$.

(b) The Bayes estimator of θ for the squared error loss function is the posterior mean of θ. Compute the Bayes estimator of θ.

(c) Explain why the posterior distribution of θ given $X_1, ..., X_n$ is the same as the posterior distribution of θ given $\sum_{i=1}^n X_i$.

Answer:

(a) The posterior density of θ

$$\begin{aligned}
\pi(\theta|x_1, \cdots, x_n) &\propto \pi(\theta) f(x_1, \cdots, x_n|\theta) \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{-\theta/\beta} \theta^{\alpha-1} \theta^n e^{-\theta \sum_{i=1}^n x_i} \\
&\propto e^{-\theta(\frac{1}{\beta} + \sum_{i=1}^n x_i)} \theta^{n+\alpha-1}.
\end{aligned}$$

So $\theta|X_1, \cdots, X_n \sim$Gamma$(n + \alpha, \frac{\beta}{n\beta\bar{x}+1})$.

(b) The Bayes estimator is

$$\hat{\theta}_{Bayes} = E(\theta|X_1, \cdots, X_n) = \frac{(n + \alpha)\beta}{n\beta\bar{X} + 1}.$$

(c) Since $\sum_{i=1}^n X_i$ is sufficient for θ, by Factorization theorem, we have

$$f(x_1, \cdots, x_n) = h(x_1, \cdots, x_n) g(\sum_{i=1}^n x_i|\theta),$$

where $g(\sum_{i=1}^n x_i|\theta)$ can be taken as the density function of $\sum_{i=1}^n X_i$ (up to a constant). Therefore,

$$\begin{aligned}
\pi(\theta|x_1, \cdots, x_n) &= \frac{\pi(\theta) f(x_1, \cdots, x_n|\theta)}{\int \pi(\theta) f(x_1, \cdots, x_n|\theta) d\theta} \\
&= \frac{\pi(\theta) h(x_1, \cdots, x_n) g(\sum_{i=1}^n x_i|\theta)}{\int \pi(\theta) h(x_1, \cdots, x_n) g(\sum_{i=1}^n x_i|\theta) d\theta} \\
&= \frac{\pi(\theta) g(\sum_{i=1}^n x_i|\theta)}{\int \pi(\theta) g(\sum_{i=1}^n x_i|\theta) d\theta} \\
&= \pi(\theta|\sum_{i=1}^n x_i).
\end{aligned}$$

This is true for any sufficient statistic.