STAT 522 **Practice Midterm Exam** **02/18/08**

1. Let $X_1, X_2, ..., X_n$ be n independent samples from $N(\mu, \sigma^2)$. Let $l_1, l_2, ..., l_n$ and $m_1, m_2, ..., m_n$ be known constants satisfying $\sum_{i=1}^n l_i = \sum_{i=1}^n m_i = 0, \sum_{i=1}^n l_i^2 = \sum_{i=1}^n m_i^2 = 1$. Define $U = \sum_{i=1}^n l_i X_i$ and $V = \sum_{i=1}^n m_i X_i$.
   - (a) State the joint distribution of U and V. Under what conditions are these two random variables independent?
   - (b) Under the condition obtained in (a), state the distribution of the following statistics:
   $$\frac{U^2}{V^2}, \quad \frac{U^2}{U^2+V^2}, \quad \frac{U}{|V|},$$
2. (a) Suppose X is a Binomial(n, p) random variable, $0 < p < 1$. Find the m.l.e. for $p(1-p)$. Show that the m.l.e. is not unbiased for $p(1-p)$. Construct an unbiased estimator for $p(1-p)$ using this m.l.e.
   - (b) Suppose X is a Binomial$(2, p)$ random variable where p can take only two values $\frac{1}{2}$ and $\frac{1}{4}$. Show that this family is not complete by constructing a non-zero function $g(X)$ whose expectation is zero for both $p = \frac{1}{2}$ and $\frac{1}{4}$.
3. Let X_1, X_2, X_3 be a random sample of size three from the uniform distribution $U(0, \theta)$, where $\theta > 0$ is an unknown parameter. Let $X_{(1)}, X_{(2)}, X_{(3)}$ be the corresponding order statistics.
   - (a) Find the marginal pdf $X_{(1)}$ and show that $X_{(1)}/\theta$ is distributed as Beta$(1, 3)$.
   - (b) Compute $E[X_{(1)}]$. Construct an unbiased estimator for θ using $X_{(1)}$.
   - (c) Show that $X_{(3)}/X_{(1)}$ is independent of $X_{(3)}$.
4. Let $X_1, X_2, ..., X_n$ be a random sample from Unif$(a\theta, b\theta)$, where $a < b$ are positive constants and $\theta > 0$ is an unknown parameter.
   - (a) Find a minimal sufficient statistic for θ.
   - (b) Is the minimal statistic found in part (a) complete? Justify your answer.
   - (c) Find the m.l.e for θ.
   - (d) Find the m.l.e for population median.
5. Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with an exponential distribution
   $$f(x|\theta) = \theta e^{-\theta x}, \quad \text{where} \quad x > 0, \theta > 0.$$
   Suppose the prior distribution of θ is Gamma(α, β).
   - (a) Find the posterior density function of θ given $X_1, ..., X_n$.
   - (b) The Bayes estimator of θ for the squared error loss function is the posterior mean of θ. Compute the Bayes estimator of θ.
   - (c) Explain why the posterior distribution of θ given $X_1, ..., X_n$ is the same as the posterior distribution of θ given $\sum_{i=1}^n X_i$.