STAT 522 Practice Final Exam - Spring 2008

1. Let X_1, X_2, X_3 be three random samples from Unif$(0, \theta)$, where $\theta > 0$ is unknown.
   (a) Show that $X_{(1)}/\theta$ is distributed as Beta$(1, 3)$.
   (b) Compute $E[X_{(1)}]$. Construct an unbiased estimator for θ using $X_{(1)}$.
   (c) Show that $X_{(3)}$ is a sufficient statistic.
   (d) Find the conditional pdf of $X_{(1)}$ given $X_{(3)}$.
   (e) Compute $W = E[4X_{(1)}|X_{(3)}]$. Show that W is unbiased for θ.
   (f) Which is better estimator for θ, W or the one constructed in (b)? Show details to justify your answer.

2. Let $X_1, \cdots, X_n$ be independent samples from $N(0, \sigma^2)$, where $\sigma > 0$ is unknown. Let

$$T = n^{-1} \sum_{i=1}^{n} X_i^2, \quad \text{and} \quad \bar{X} = n^{-1} \sum_{i=1}^{n} X_i.$$

   (a) Show that T is the UMVUE for σ^2.
   (b) Is T a consistent estimator of σ^2? Justify your answer.
   (c) Obtain the Cramér-Rao lower bound for the variance of unbiased estimators of σ^2. Is this bound attainable?
   (d) Describe the distribution of T.
   (e) Show that $\bar{X}/\sqrt{T}$ is ancillary and is independent of T.
   (f) Find the MLE of σ^2 and the asymptotic distribution of the estimator.
   (g) Find the MLE of σ^{-2} and the asymptotic distribution of the estimator.
   (h) Give the LRT for $H_0 : \sigma^2 = 1$ vs $H_1 : \sigma^2 \neq 1$. Simplify your answer at your best.

3. Let $X_1, ..., X_n$ be a random sample from Poisson(λ). We want to estimate $e^{-\lambda}$.
   (a) Define the function

$$\begin{aligned} U(X_1) &= 1, \quad \text{if} \quad X_1 = 0; \\ &= 0, \quad \text{if} \quad X_1 \neq 0. \end{aligned}$$

   Find the UMVU estimator of $e^{-\lambda}$ by using Rao-Blackwell theorem.
   (b) Compute the Cramér-Rao lower variance bound for unbiased estimators of $e^{-\lambda}$.
   (c) Find the MLE of $e^{-\lambda}$.
   (d) Using Delta method, show that the approximate large sample variance of the MLE of $e^{-\lambda}$ is same as the Cramér-Rao Lower Bound given in (b).

4. Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with the common pdf

$$f(x|\theta) = \frac{1}{2\theta^3} e^{-\frac{x}{\theta}} x^2, \quad \text{where} \quad x > 0, \quad \theta > 0.$$

   (a) Show that this family has the monotone likelihood ratio property.
   (b) Find the UMP test procedure for $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$ at the level $\alpha = 0.05$. Explain how you can use χ^2 test to find the critical (rejection) region.
   (c) Compute the power function of the test procedure obtained in (b).

5. Let $X_1, ..., X_n$ be n independent observations from $N(\mu, 1)$.

   (a) Find the likelihood ratio test procedure for $H_0 : \mu \leq 2$ vs $H_1 : \mu > 2$ at size $\alpha = 0.05$.
   (b) Calculate the power function for the LRT test. Is the power function increasing or decreasing? Justify your answer.
   (c) How large should n be so that the test in (a) has power 0.9 for $\mu = \mu_0 + 0.5$?
   (d) If we took 9 samples and observed $\bar{x} = 2.5$. Compute the p-value based on the data for the test in (a). Will you reject H_0 at the level $\alpha = 0.05$ based on this p-value? Explain your answer.

   (You may need the following information:

$$z_{0.1} = 1.28, z_{0.05} = 1.645, z_{0.025} = 1.960, z_{0.01} = 2.33, z_{0.005} = 2.58.)$$

6. Suppose $X_1, X_2, ..., X_n$ is a random sample from $N(\mu, \sigma^2)$, where σ^2 is unknown.

   (a) Using the pivotal method, set up a $100(1-\alpha)\%$ confidence interval of μ.
   (b) Show that the shortest confidence interval (with minimal expected length) constructed in (a) is equal-tailed.