ST 522, Spring 2008

HW 5 Solutions

6.31

c. (i)

$$\mathbf{E}(X^k) = \mathrm{E}\left(\frac{X}{Y}Y\right)^k = \mathrm{E}\left[\left(\frac{X}{Y}\right)^k (Y^k)\right] \overset{\text{indep.}}{=} \mathrm{E}\left(\frac{X}{Y}\right)^k \mathrm{E}\left(Y^k\right).$$

Divide both sides by $\mathrm{E}\left(Y^k\right)$ to obtain the desired equality.

(ii) If α is fixed, $T = \sum_i X_i$ is a complete sufficient statistic for β by Theorem 6.2.25. Because β is a scale parameter, if $Z_1, \ldots, Z_n$ is a random sample from a gamma$(\alpha, 1)$ distribution, then $X_{(i)}/T$ has the same distribution as $(\beta Z_{(i)})/\left(\beta \sum_i Z_i\right) = Z_{(i)}/\left(\sum_i Z_i\right)$, and this distribution does not depend on β. Thus, $X_{(i)}/T$ is ancillary, and by Basu's Theorem, it is independent of T. We have

$$\mathrm{E}(X_{(i)}|T) = \mathrm{E}\left(\frac{X_{(i)}}{T}T\middle| T\right) = T\mathrm{E}\left(\frac{X_{(i)}}{T}\middle| T\right) \overset{\text{indep.}}{=} T\mathrm{E}\left(\frac{X_{(i)}}{T}\right) \overset{\text{part (i)}}{=} T\frac{\mathrm{E}(X_{(i)})}{\mathrm{E}T}.$$

Note, this expression is correct for each fixed value of (α, β), regardless whether α is "known" or not.

7.1 For each value of x, the MLE $\hat{\theta}$ is the value of θ that maximizes $f(x|\theta)$. These values are in the following table.

x	0	1	2	3	4
$\hat{\theta}$	1	1	2 or 3	3	3

At $x = 2$, $f(x|2) = f(x|3) = 1/4$ are both maxima, so both $\hat{\theta} = 2$ or $\hat{\theta} = 3$ are MLEs.

7.2 a.

$$\begin{aligned} L(\beta|x) &= \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^\alpha} x_i^{\alpha-1} e^{-x_i/\beta} = \frac{1}{\Gamma(\alpha)^n \beta^{n\alpha}} \left[\prod_{i=1}^{n} x_i\right]^{\alpha-1} e^{-\Sigma_i x_i/\beta} \\ \log L(\beta|x) &= -\log\Gamma(\alpha)^n - n\alpha\log\beta + (\alpha-1)\log\left[\prod_{i=1}^{n} x_i\right] - \frac{\sum_i x_i}{\beta} \\ \frac{\partial \log L}{\partial\beta} &= -\frac{n\alpha}{\beta} + \frac{\sum_i x_i}{\beta^2} \end{aligned}$$

Set the partial derivative equal to 0 and solve for β to obtain $\hat{\beta} = \sum_i x_i/(n\alpha)$. To check that this is a maximum, calculate

$$\left.\frac{\partial^2 \log L}{\partial \beta^2}\right|_{\beta=\hat{\beta}} = \left.\frac{n\alpha}{\beta^2} - \frac{2\sum_i x_i}{\beta^3}\right|_{\beta=\hat{\beta}} = \frac{(n\alpha)^3}{\left(\sum_i x_i\right)^2} - \frac{2(n\alpha)^3}{\left(\sum_i x_i\right)^2} = -\frac{(n\alpha)^3}{\left(\sum_i x_i\right)^2} < 0.$$

Because $\hat{\beta}$ is the unique point where the derivative is 0 and it is a local maximum, it is a global maximum. That is, $\hat{\beta}$ is the MLE.

7.6 a. $f(\mathbf{x}|\theta) = \prod_i \theta x_i^{-2} I_{[\theta,\infty)}(x_i) = \left(\prod_i x_i^{-2}\right) \theta^n I_{[\theta,\infty)}(x_{(1)})$. Thus, $X_{(1)}$ is a sufficient statistic for θ by the Factorization Theorem.

c. $\mathrm{E}X = \int_\theta^\infty \theta x^{-1}\, dx = \theta \log x|_\theta^\infty = \infty$. Thus the method of moments estimator of θ does not exist. (This is the Pareto distribution with $\alpha = \theta$, $\beta = 1$.)

7.7 $L(0|\mathbf{x}) = 1$, $0 < x_i < 1$, and $L(1|\mathbf{x}) = \prod_i 1/(2\sqrt{x_i})$, $0 < x_i < 1$. Thus, the MLE is 0 if $1 \geq \prod_i 1/(2\sqrt{x_i})$, and the MLE is 1 if $1 < \prod_i 1/(2\sqrt{x_i})$.

7.8 a. $\mathrm{E}X^2 = \mathrm{Var}\, X + \mu^2 = \sigma^2$. Therefore X^2 is an unbiased estimator of σ^2.

b.

$$
\begin{aligned}
L(\sigma|\mathbf{x}) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)}. \quad \log L(\sigma|\mathbf{x}) = \log(2\pi)^{-1/2} - \log\sigma - x^2/(2\sigma^2). \\
\frac{\partial \log L}{\partial\sigma} &= -\frac{1}{\sigma} + \frac{x^2}{\sigma^3} \stackrel{\text{set}}{=} 0 \Rightarrow \hat\sigma X^2 = \hat\sigma^3 \;\Rightarrow \hat\sigma = \sqrt{X^2} = |X|. \\
\frac{\partial^2 \log L}{\partial\sigma^2} &= \frac{-3x^2\sigma^2}{\sigma^6} + \frac{1}{\sigma^2}, \text{ which is negative at } \hat\sigma = |x|.
\end{aligned}
$$

Thus, $\hat\sigma = |x|$ is a local maximum. Because it is the only place where the first derivative is zero, it is also a global maximum.

c. Because $\mathrm{E}X = 0$ is known, just equate $\mathrm{E}X^2 = \sigma^2 = \frac{1}{n}\sum_{i=1}^1 X_i^2 = X^2 \Rightarrow \hat\sigma = |X|$.

7.11 a.

$$
\begin{aligned}
f(\mathbf{x}|\theta) &= \prod_i \theta x_i^{\theta-1} = \theta^n \left(\prod_i x_i\right)^{\theta-1} = L(\theta|\mathbf{x}) \\
\frac{d}{d\theta}\log L &= \frac{d}{d\theta}\left[n\log\theta + (\theta-1)\log\prod_i x_i\right] = \frac{n}{\theta} + \sum_i \log x_i.
\end{aligned}
$$

Set the derivative equal to zero and solve for θ to obtain $\hat\theta = (-\frac{1}{n}\sum_i \log x_i)^{-1}$. The second derivative is $-n/\theta^2 < 0$, so this is the MLE. To calculate the variance of $\hat\theta$, note that $Y_i = -\log X_i \sim \text{exponential}(1/\theta)$, so $-\sum_i \log X_i \sim \text{gamma}(n, 1/\theta)$. Thus $\hat\theta = n/T$, where $T \sim \text{gamma}(n, 1/\theta)$. We can either calculate the first and second moments directly, or use the fact that $\hat\theta$ is inverted gamma (page 51). We have

$$
\begin{aligned}
\mathrm{E}\frac{1}{T} &= \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{t} t^{n-1} e^{-\theta t}\, dt = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-1)}{\theta^{n-1}} = \frac{\theta}{n-1}. \\
\mathrm{E}\frac{1}{T^2} &= \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{t^2} t^{n-1} e^{-\theta t}\, dt = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-2)}{\theta^{n-2}} = \frac{\theta^2}{(n-1)(n-2)},
\end{aligned}
$$

and thus

$$
\mathrm{E}\,\hat\theta = \frac{n}{n-1}\theta \quad \text{and} \quad \mathrm{Var}\,\hat\theta = \frac{n^2}{(n-1)^2(n-2)}\theta^2 \to 0 \text{ as } n \to \infty.
$$

b. Because $X \sim \text{beta}(\theta, 1)$, $\mathrm{E}X = \theta/(\theta+1)$ and the method of moments estimator is the solution to

$$
\frac{1}{n}\sum_i X_i = \frac{\theta}{\theta+1} \Rightarrow \tilde\theta = \frac{\sum_i X_i}{n - \sum_i X_i}.
$$