

## HW 1 Solutions

5.21 Let  $m$  denote the median. Then, for general  $n$  we have

$$\begin{aligned} P(\max(X_1, \dots, X_n) > m) &= 1 - P(X_i \leq m \text{ for } i = 1, 2, \dots, n) \\ &= 1 - [P(X_1 \leq m)]^n = 1 - \left(\frac{1}{2}\right)^n. \end{aligned}$$

5.22 Calculating the cdf of  $Z^2$ , we obtain

$$\begin{aligned} F_{Z^2}(z) &= P((\min(X, Y))^2 \leq z) = P(-\sqrt{z} \leq \min(X, Y) \leq \sqrt{z}) \\ &= P(\min(X, Y) \leq \sqrt{z}) - P(\min(X, Y) \leq -\sqrt{z}) \\ &= [1 - P(\min(X, Y) > \sqrt{z})] - [1 - P(\min(X, Y) > -\sqrt{z})] \\ &= P(\min(X, Y) > -\sqrt{z}) - P(\min(X, Y) > \sqrt{z}) \\ &= P(X > -\sqrt{z})P(Y > -\sqrt{z}) - P(X > \sqrt{z})P(Y > \sqrt{z}), \end{aligned}$$

where we use the independence of  $X$  and  $Y$ . Since  $X$  and  $Y$  are identically distributed,  $P(X > a) = P(Y > a) = 1 - F_X(a)$ , so

$$F_{Z^2}(z) = (1 - F_X(-\sqrt{z}))^2 - (1 - F_X(\sqrt{z}))^2 = 1 - 2F_X(-\sqrt{z}),$$

since  $1 - F_X(\sqrt{z}) = F_X(-\sqrt{z})$ . Differentiating and substituting gives

$$f_{Z^2}(z) = \frac{d}{dz} F_{Z^2}(z) = f_X(-\sqrt{z}) \frac{1}{\sqrt{z}} = \frac{1}{\sqrt{2\pi}} e^{-z/2} z^{-1/2},$$

the pdf of a  $\chi_1^2$  random variable. Alternatively,

$$\begin{aligned} P(Z^2 \leq z) &= P([\min(X, Y)]^2 \leq z) \\ &= P(-\sqrt{z} \leq \min(X, Y) \leq \sqrt{z}) \\ &= P(-\sqrt{z} \leq X \leq \sqrt{z}, X \leq Y) + P(-\sqrt{z} \leq Y \leq \sqrt{z}, Y \leq X) \\ &= P(-\sqrt{z} \leq X \leq \sqrt{z} | X \leq Y)P(X \leq Y) \\ &\quad + P(-\sqrt{z} \leq Y \leq \sqrt{z} | Y \leq X)P(Y \leq X) \\ &= \frac{1}{2}P(-\sqrt{z} \leq X \leq \sqrt{z}) + \frac{1}{2}P(-\sqrt{z} \leq Y \leq \sqrt{z}), \end{aligned}$$

using the facts that  $X$  and  $Y$  are independent, and  $P(Y \leq X) = P(X \leq Y) = \frac{1}{2}$ . Moreover, since  $X$  and  $Y$  are identically distributed

$$P(Z^2 \leq z) = P(-\sqrt{z} \leq X \leq \sqrt{z})$$

and

$$\begin{aligned} f_{Z^2}(z) &= \frac{d}{dz} P(-\sqrt{z} \leq X \leq \sqrt{z}) = \frac{1}{\sqrt{2\pi}} (e^{-z/2} \frac{1}{2} z^{-1/2} + e^{-z/2} \frac{1}{2} z^{-1/2}) \\ &= \frac{1}{\sqrt{2\pi}} z^{-1/2} e^{-z/2}, \end{aligned}$$

the pdf of a  $\chi_1^2$ .

$$\begin{aligned}
P(Z > z) &= \sum_{x=1}^{\infty} P(Z > z|x)P(X = x) = \sum_{x=1}^{\infty} P(U_1 > z, \dots, U_x > z|x)P(X = x) \\
&= \sum_{x=1}^{\infty} \prod_{i=1}^x P(U_i > z)P(X = x) \quad (\text{by independence of the } U_i\text{'s}) \\
&= \sum_{x=1}^{\infty} P(U_i > z)^x P(X = x) = \sum_{x=1}^{\infty} (1-z)^x \frac{1}{(e-1)x!} \\
&= \frac{1}{(e-1)} \sum_{x=1}^{\infty} \frac{(1-z)^x}{x!} = \frac{e^{1-z} - 1}{e-1} \quad 0 < z < 1.
\end{aligned}$$

5.24 Use  $f_X(x) = 1/\theta$ ,  $F_X(x) = x/\theta$ ,  $0 < x < \theta$ . Let  $Y = X_{(n)}$ ,  $Z = X_{(1)}$ . Then, from Theorem 5.4.6,

$$f_{Z,Y}(z, y) = \frac{n!}{0!(n-2)!0!} \frac{1}{\theta} \frac{1}{\theta} \left(\frac{z}{\theta}\right)^0 \left(\frac{y-z}{\theta}\right)^{n-2} \left(1-\frac{y}{\theta}\right)^0 = \frac{n(n-1)}{\theta^n} (y-z)^{n-2}, \quad 0 < z < y < \theta.$$

Now let  $W = Z/Y$ ,  $Q = Y$ . Then  $Y = Q$ ,  $Z = WQ$ , and  $|J| = q$ . Therefore

$$f_{W,Q}(w, q) = \frac{n(n-1)}{\theta^n} (q - wq)^{n-2} q = \frac{n(n-1)}{\theta^n} (1-w)^{n-2} q^{n-1}, \quad 0 < w < 1, 0 < q < \theta.$$

The joint pdf factors into functions of  $w$  and  $q$ , and, hence,  $W$  and  $Q$  are independent.

5.27 a.  $f_{X_{(i)}|X_{(j)}}(u|v) = f_{X_{(i)}, X_{(j)}}(u, v)/f_{X_{(j)}}(v)$ . Consider two cases, depending on which of  $i$  or  $j$  is greater. Using the formulas from Theorems 5.4.4 and 5.4.6, and after cancellation, we obtain the following.

(i) If  $i < j$ ,

$$\begin{aligned}
f_{X_{(i)}|X_{(j)}}(u|v) &= \frac{(j-1)!}{(i-1)!(j-1-i)!} f_X(u) F_X^{i-1}(u) [F_X(v) - F_X(u)]^{j-i-1} F_X^{1-j}(v) \\
&= \frac{(j-1)!}{(i-1)!(j-1-i)!} \frac{f_X(u)}{F_X(v)} \left[\frac{F_X(u)}{F_X(v)}\right]^{i-1} \left[1 - \frac{F_X(u)}{F_X(v)}\right]^{j-i-1}, \quad u < v.
\end{aligned}$$

Note this interpretation. This is the pdf of the  $i$ th order statistic from a sample of size  $j-1$ , from a population with pdf given by the truncated distribution,  $f(u) = f_X(u)/F_X(v)$ ,  $u < v$ .

(ii) If  $j < i$  and  $u > v$ ,

$$\begin{aligned}
f_{X_{(i)}|X_{(j)}}(u|v) &= \frac{(n-j)!}{(n-1)!(i-1-j)!} f_X(u) [1-F_X(u)]^{n-i} [F_X(u) - F_X(v)]^{i-1-j} [1-F_X(v)]^{j-n} \\
&= \frac{(n-j)!}{(i-j-1)!(n-i)!} \frac{f_X(u)}{1-F_X(v)} \left[\frac{F_X(u) - F_X(v)}{1-F_X(v)}\right]^{i-j-1} \left[1 - \frac{F_X(u) - F_X(v)}{1-F_X(v)}\right]^{n-i}.
\end{aligned}$$

This is the pdf of the  $(i-j)$ th order statistic from a sample of size  $n-j$ , from a population with pdf given by the truncated distribution,  $f(u) = f_X(u)/(1-F_X(v))$ ,  $u > v$ .

6.2 By the Factorization Theorem,  $T(X) = \min_i(X_i/i)$  is sufficient because the joint pdf is

$$f(x_1, \dots, x_n; \theta) = \prod_{i=1}^n e^{i\theta - x_i} I\{x_i \geq i\theta\} = e^{-\sum_i x_i} I\{T(x) \geq \theta\} \prod_{i=1}^n e^{i\theta}$$

Notice, we use the fact that  $i > 0$ , and the fact that all  $x_i$ s  $> i\theta$  if and only if  $\min_i(x_i/i) > \theta$ .