Chapter 9: Interval Estimation

1 Why Interval Estimators

Interval estimator: $[L(\mathbf{X}), U(\mathbf{X})]$

Three types of intervals:

- two-sided interval $[L(\mathbf{X}), U(\mathbf{X})]$
- $[L(\mathbf{X}), \infty)$ (call $L(\mathbf{X})$ the lower confidence bound)
- $(-\infty, U(\mathbf{X})]$ (call $U(\mathbf{X})$ the upper confidence bound)

Example: $X_1, \ldots, X_4$ iid $N(\mu, 1)$. Compare two types of estimators.
(i) The point estimator of μ: $\bar{X}$

(ii) The interval estimator of $\mu : [\bar{X} - 1, \bar{X} + 1]$

Remark:

By using the interval estimator, we give up some precision in our estimate, but gain confidence or assurance about our assertion.

Definition 1: Given an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of parameter θ, its *coverage probability* is defined as

$$P(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$$

Definition 2: Given an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of parameter θ, its *confidence coefficient* is defined as

$$\min_{\theta} P(\theta \in [L(\mathbf{X}), U(\mathbf{X})]).$$

Definition 3: Given an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of parameter θ, if its confidence coefficient is $1-\alpha$, we call it a $(1-\alpha)$ *confidence interval* or *confidence set*.

Definition 4: Given an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of parameter θ, its *expected length* is defined as

$$E_{\theta}[U(\mathbf{X}) - L(\mathbf{X})]$$

Example: Assume $X_1, \cdots, X_n \text{Unif}(0, \theta)$. Let $Y = X_{(n)}$. Using Y, we construct two $(1-\alpha)$ confidence intervals.

(1) Interval $[aY, bY]$.

(2) Interval $[Y + c, Y + d]$.

How to Construct Confidence Intervals?
Method 1: By inverting the acceptance region of tests;
Method 2: Using pivotal quantities.

2 Inverting Test

Remark: Both hypotheses testing and CI look for consistency between samples and parameters, but from slightly different perspective.

- **Hypothesis**: Fix the parameter — asks what sample values (in the appropriate region) are consistent with that fixed value.
- **Confidence set**: Fix the sample value — asks what parameter values make this sample most plausible.

There is one-to-one correspondence between tests and confidence intervals.

Example: $X_1, ..., X_n$ iid $N(\theta, \sigma^2)$, σ known. Test or CI for θ.

THEOREM:

(1) For each $\theta_0 \in \Theta$, let $A(\theta_0)$ be the acceptance region of a level α test $H_0 : \theta = \theta_0$. Define a set $C(\mathbf{x}) = \{\theta_0 : \mathbf{x} \in A(\theta_0)\}$. Then the random set $C(\mathbf{X})$ is a $(1-\alpha)$-confidence set.

(2) Conversely, if $C(\mathbf{x})$ is a $(1-\alpha)$ confidence set for θ, for any θ_0, define the acceptance region of a test for the hypothesis $H_0 : \theta = \theta_0$ by $A(\theta_0) = \{\mathbf{x} : \theta_0 \in C(\mathbf{x})\}$. Then the test has level α.

Proof: (see textbook)

Note: $A(\theta_0)$ is a set in the *sample space*, and $C(\mathbf{x})$ is a set in the *parameter space*. In the above normal example,

$$A(\theta_0) = \{(x_1, \cdots, x_n) : \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \bar{x} \le \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\}$$

$$C(\mathbf{x}) = \{\mu_0 : \bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu_0 \le \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\}.$$

The $(1-\alpha)$ confidence interval is $C(\mathbf{X}) = \left[\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$.

Method 1: Inverting Test Statistic

We can invert an acceptance region of a test to get a confidence interval.

- To obtain a $(1-\alpha)$ two-sided confidence interval $[L(\mathbf{X}), U(\mathbf{X})]$, we invert the acception region of a level α test for

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta \neq \theta_0$$

- To obtain a $(1-\alpha)$ one-sided confidence interval $[L(\mathbf{X}), \infty)$, we invert the acception region of a level α test for

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta > \theta_0$$

- To obtain a $(1-\alpha)$ one-sided confidence interval $(-\infty, U(\mathbf{X})]$, we invert the acception region of a level α test for

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta < \theta_0$$

Example: $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$, both unknown.

(i) Find $1 - \alpha$ confidence interval for μ.

(ii) Find $1 - \alpha$ upper confidence bound for μ.

(iii) Find $1 - \alpha$ lower confidence bound for μ.

Inverting LRT

Example. $X_1, \ldots, X_n \sim \text{Exp}(\lambda)$.

Example: (Discrete)

$X_1, \ldots, X_n \sim \text{Bin}(1, p)$. Find lower confidence bound.

Fact: Assume T has an MLR. Let $c_1(\theta)$ and $c_2(\theta)$ be the cut-off values satisfying

$$P_\theta(T(\mathbf{X}) > c_2(\theta)) = \alpha, \quad \text{and} \quad P_\theta(T(\mathbf{X}) < c_1(\theta)) = \alpha.$$

Then both $c_1(\theta)$ and $c_2(\theta)$ are increasing in θ.

3 Pivotal Quantity

A random quantity $Q(\mathbf{X}, \theta)$ is called *pivotal* (or a *pivot*) if the distribution of $Q(\mathbf{X}, \theta)$ is independent of θ.

Note: this is different from an ancillary statistic since $Q(\mathbf{X}, \theta)$ also depends on θ and hence is not a statistic.

- location family: iid $f(x - \theta)$.

$$\bar{X} - \theta$$

- scale family: iid $\frac{1}{\sigma} f(x/\sigma)$.

$$\bar{X}/\sigma$$

- location-scale family: iid $\frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$.

$$\frac{\bar{X} - \mu}{S}$$

Example: $X_1, \ldots, X_n \sim \exp(\lambda)$. Show the following are pivotal. What distributions do they have?
(1) $\frac{X_1}{\lambda}$
(2) $\sum_{i=1}^n X_i/\lambda$
(3) $2\sum_{i=1}^n X_i/\lambda$

THEOREM: If the pdf of T is expressed as

$$f(t|\theta) = g(Q(t,\theta))|(\partial/\partial t)Q(t,\theta)|, \quad \text{and } Q \text{ is monotone in t,}$$

then $Q(T,\theta)$ is a pivot.

Example: Let $X_1, \ldots, X_n \sim \exp(\lambda)$. Show $Q = \sum_{i=1}^n X_i/\lambda$ is pivotal.

Construct confidence set with pivotal quantity.

(i) Find a, b such that $P_\theta(a \le Q(X,\theta) \le b) = 1-\alpha$.

(ii) Define $C(\mathbf{x}) = \{\theta : a \le Q(\mathbf{x},\theta) \le b\}$.

$$P_\theta(\theta \in C(\mathbf{X})) = P_\theta(a \le Q(\mathbf{X},\theta) \le b) = 1-\alpha.$$

When will $C(\mathbf{X})$ be an interval? **Answer**: If $Q(\mathbf{x},\theta)$ is monotone in θ, then $C(\mathbf{X})$ is an interval.

- If $Q(\mathbf{x},\theta)$ is increasing in θ, then
$$C(\mathbf{x}) = \{\theta : L(\mathbf{x},a) \le \theta \le U(\mathbf{x},b)\}.$$
- If $Q(\mathbf{x},\theta)$ is decreasing in θ, then
$$C(\mathbf{x}) = \{\theta : L(\mathbf{x},b) \le \theta \le U(\mathbf{x},a)\}.$$

Example. iid exponential.

Normal Pivot for μ

Example. iid $N(\mu, \sigma^2)$, σ known. Interval for μ.

Example. iid $N(\mu, \sigma^2)$, σ unknown. Interval for μ.

Normal Pivot for σ

Example. iid $N(\mu, \sigma^2)$, μ known. Interval for σ.

Example. iid $N(\mu, \sigma^2)$, μ unknown. Interval for σ.

Pivoting the CDFs

If T is statistic, its cdf $F(t|\theta) = P_\theta(T \le t)$ follows

$$P_\theta\left(F(T|\theta) \le u\right) = P_\theta\left(T \le F^{-1}(u)\right) = F[F^{-1}(u)] = u,$$

where $F(t|\theta) \in [0,1]$ and is increasing in t.

- Define the function $Y = F(T|\theta)$. Then $Y \sim U(0,1)$ and it is a pivot.

Given α, we can choose α_1, α_2 such that

$$P\left(\alpha_1 \le F(T|\theta) \le 1-\alpha_2\right) = 1-\alpha, \quad \text{with} \quad \alpha_1 + \alpha_2 = \alpha.$$

- If $F(t|\theta)$ is decreasing in θ for all t, define θ_L, θ_U by

  $$F(t|\theta_L) = 1-\alpha_2, \quad F(t|\theta_U) = \alpha_1.$$

  Then $[\theta_L(T), \theta_U(T)]$ is $(1-\alpha)$ CI for θ.

- If $F(t|\theta)$ is increasing in θ for all t, define θ_L, θ_U by

  $$F(t|\theta_L) = \alpha_1, \quad F(t|\theta_U) = 1-\alpha_2.$$

  Then $[\theta_L(T), \theta_U(T)]$ is $(1-\alpha)$ CI for θ.

Example. iid from $f(x|\mu) = e^{-(x-\mu)}I(x > \mu)$. $T = X_{(1)}$ sufficient. Construct the $100(1-\alpha)\%$CI for μ by pivoting the cdf of T.

4 Evaluate and Compare Interval Estimators

- coverage probabilities
- expected length

Shortest Expected Length: Given α, the $(1-\alpha)$ CIs are not unique. Among many choices, we want to minimize expected length

$$\min E\left[U(\mathbf{X}) - L(\mathbf{X})\right].$$

Example: (Optimizing the length) Let $X_1, \cdots, X_n$ be iid $N(\mu, \sigma^2)$, where σ^2 is known. Using the pivotal

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}},$$

we choose a and b such that

$$P(a \leq Z \leq b) = 1 - \alpha.$$

The $(1-\alpha)$ confidence interval is given by $\left[\bar{X} - b\frac{\sigma}{\sqrt{n}}, \bar{X} - a\frac{\sigma}{\sqrt{n}}\right]$.

(1) What is the length of the interval?

(2) How to choose a and b?

How to optimize the length?

- Case 1: Unimodal case
- Case 2: Lagrange Method

Theorem 9.3.2 Let $f(x)$ be a unimodal pdf. If the interval $[a, b]$ satisfies

(i) $\int_a^b f(x)dx = 1 - \alpha$.

(ii) $f(a) = f(b) > 0$ and $a \leq x^* \leq b$, where x^* is a mode of $f(x)$.

Then $[a, b]$ is the shortest among all intervals that satisfies (i).
Proof: See textbook.

Example. iid $N(\mu, \sigma^2)$ σ known.

Example. iid $N(\mu, \sigma^2)$ σ unknown.

Lagrange Multiplier Method

$$\begin{aligned} \text{Minimize} \quad & b - a \\ \text{subject to} \quad & \int_a^b f(x)dx = 1 - \alpha. \end{aligned}$$

Example. iid $N(\mu, \sigma^2)$ σ known.

Example. iid exponential.