ST521-002 Homework #1 Solution
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1.1
(a) $S = \{TTTT, HTTT, THTT, TTHT, TTTH, HHTT, HTHT, HTTH, THHT,$
$THTH, TTHH, HHHT, HHTH, HTHH, THHH, HHHH\}$
(b) $S = \{0, 1, 2, \ldots\}$
(c) $S = [0, \infty)$
(d) $S = (0, \infty)$
(e) $S = \{\frac{n}{N} : n = 0, 1, 2, \ldots, N\}$. N: the number of total parts in this shipment

1.5
(a) The event $A \cap B \cap C$ means a U.S. birth result in identical female twins
(b) $P(A \cap B \cap C) = \frac{1}{90} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{540}$

1.6
No, since

$$\begin{aligned} p_0 &= (1-u)(1-w) \\ p_1 &= u(1-w) + (1-u)w \\ p_2 &= u \cdot w \end{aligned}$$

To meet the condition, $p_0 = p_1 = p_2$, we have the following simultaneous equations
$$\begin{cases} 1 - u - w + uw = u + w - 2uw \\ 1 - u - w + uw = uw \end{cases}$$
However, there is no real root for this system.

1.7

$$\begin{aligned} P(\text{scoring 0 point}) &= P(\text{not hit the board}) = \frac{A - \pi r^2}{A} \\ P(\text{scoring } i \text{ points}) &= P(\text{hit the board} \cap \text{hit region } i) \\ &= \frac{\pi r^2}{A}\left[\frac{1}{\pi r^2}\left(\frac{(6-i)}{5}r\right)^2\pi - \frac{1}{\pi r^2}\left(\frac{(5-i)}{5}r\right)^2\pi\right] \\ &= \frac{\pi r^2}{A}\left[\frac{(6-i)^2 - (5-i)^2}{5^2}\right] \end{aligned}$$

1.13
No. Argue by negation.
Suppose A and B are disjoint, then $P(A \cup B) = P(A) + P(B) = P(A) + [1 - P(B^c)] = \frac{1}{3} + [1 - \frac{1}{4}] > 1$.
Thus, A and B cannot be disjoint.

1.24
(a)
$$\begin{aligned} P(\text{A wins}) &= P(\text{H}) + P(\text{TTH}) + P(\text{TTTTH}) + \cdots \\ &= \tfrac{1}{2} + \tfrac{1}{4}\tfrac{1}{2} + \left(\tfrac{1}{4}\right)^2 \tfrac{1}{2} \\ &= \frac{\frac{1}{2}}{1 - \frac{1}{4}} \\ &= \tfrac{2}{3} \end{aligned}$$

(b)

$$
\begin{aligned}
P(\text{A wins}) &= p + (1-p)^2 p + (1-p)^4 p + \cdots \\
&= \tfrac{p}{1-(1-p)^2} \\
&= \tfrac{1}{2-p}
\end{aligned}
$$

(c)

Since the probability of A wins is $\frac{1}{2-p}$, the probability of A wins will be minimized when p approximates 0. In that case, the probability of A wins is $\frac{1}{2}$. Therefore, for any $p \in (0,1)$, the probability of A wins will be greater than a half.

1.27

(a)

$$\sum_{k=0}^{n}(-1)^k C_k^n = \sum_{k=0}^{n} C_k^n (-1)^k 1^{n-k} = [(-1)+1]^n = 0$$

(b)

$$
\begin{aligned}
\sum_{k=1}^{n} k C_k^n = \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} &= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} \\
&= n \sum_{k=1}^{n} C_{k-1}^{n-1} \\
&= n \sum_{k'=0}^{n-1} C_{k'}^{n-1} 1^{k'} 1^{n-1-k'} \\
&= n 2^{n-1}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\sum_{k=1}^{n}(-1)^{k+1} k C_k^n = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!}(-1)^{k-1} &= n \sum_{k=1}^{n} C_{k-1}^{n-1}(-1)^{k-1} \\
&= n \sum_{k'=1}^{n-1} C_{k'}^{n-1}(-1)^{k'} 1^{n-1-k'} \\
&= [(-1)+1]^{n-1} = 0
\end{aligned}
$$

1.34

(a) P(brown-haired animal is chosen)$=\frac{1}{2}\frac{2}{3} + \frac{1}{2}\frac{3}{5} = \frac{19}{30}$

(b) P(From litter 1|brown-haired animal is chosen) $= \frac{\frac{1}{3}}{\frac{19}{30}} = \frac{10}{19}$