ST521-002 Homework #11 Solution
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4.39
Define a set $\mathcal{B}_1 = \{x_k, k \neq i, j : \sum_{k\neq i,j} x_k = m - x_i - x_j\}$
The joint distribution of (X_i, X_j) is given by

$$\begin{aligned}
f(x_i, x_j) &= \sum_{\{x_k \in \mathcal{B}, k\neq i,j\}} \frac{m!}{x_1!x_2!\cdots x_i!\cdots x_j!\cdots x_n!} p_1^{x_1} p_2^{x_2} \cdots p_i^{x_i} \cdots p_j^{x_j} \cdots p_n^{x_n} \\
&= \frac{1}{x_i!x_j!} p_i^{x_i} p_j^{x_j} \sum_{\{x_k \in \mathcal{B}, k\neq i,j\}} \frac{m!}{x_1!x_2!\cdots x_n!} p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n} \cdot \frac{(m-x_i-x_j)!(1-p_i-p_j)^{m-x_i-x_j}}{(m-x_i-x_j)!(1-p_i-p_j)^{m-x_i-x_j}} \\
&= \frac{m!}{x_i!x_j!(m-x_i-x_j)!} p_i^{x_i} p_j^{x_j} (1-p_i-p_j)^{m-x_i-x_j} \\
&\quad \sum_{\{x_k \in \mathcal{B}, k\neq i,j\}} \frac{(m-x_i-x_j)!}{x_1!x_2!\cdots x_n!} \left(\frac{p_1}{1-p_i-p_j}\right)^{x_1} \left(\frac{p_2}{1-p_i-p_j}\right)^{x_2} \cdots \left(\frac{p_n}{1-p_i-p_j}\right)^{x_n} \\
&= \frac{m!}{x_i!x_j!(m-x_i-x_j)!} p_i^{x_i} p_j^{x_j} (1-p_i-p_j)^{m-x_i-x_j}
\end{aligned}$$

The forth equality holds by using Theorem 4.6.4.
The joint distribution of (X_i, X_j) ~Multinomial(m, p_i, p_j)
By using the same technique, we can find marginally X_j will follow a binomial(m, p_j)
By defining another set $\mathcal{B}_2 = \{x_k, k \neq j : \sum_{k\neq j} x_k = m - x_j\}$ and using the similar technique, we can show that the marginal distribution of X_j ~Binomial(m, p_j)
The conditional distribution of $X_i|X_j$ is given as

$$\begin{aligned}
f(x_i|x_j) &= \frac{\frac{m!}{x_i!x_j!(m-x_i-x_j)!} p_i^{x_i} p_j^{x_j} (1-p_i-p_j)^{m-x_i-x_j}}{\frac{m!}{x_j!(m-x_j)!} p_j^{x_j} (1-p_j)^{m-x_j}} \\
&= \frac{(m-x_j)!}{x_i!(m-x_i-x_j)!} p_i^{x_i} \frac{(1-p_i-p_j)^{m-x_i-x_j}}{(1-p_j)^{m-x_j}} \\
&= C_{x_i}^{m-x_j} \left(\frac{p_i}{1-p_j}\right)^{x_i} \left(1 - \frac{p_i}{1-p_j}\right)^{m-x_i-x_j}
\end{aligned}$$

Thus, the conditional distribution of X_i given on X_j is binomial$(m - x_j, \frac{p_i}{1-p_j})$
(ii)

$$\begin{aligned}
M_{\boldsymbol{X}}(\boldsymbol{t}) = E\left(e^{t_1X_1+t_2X_2+\ldots+t_nX_n}\right) &= \sum_{X\in\mathcal{X}} e^{t_1X_1+t_2X_2+\ldots+t_nX_n} \frac{m!}{X_1!\ldots X_n!} p_1^{X_1} \ldots p_n^{X_n} \\
&= \sum_{X\in\mathcal{X}} \frac{m!}{X_1!\ldots X_n!} (p_1e^{t_1})^{X_1} \ldots (p_ne^{t_n})^{X_n} \\
&= \left[\sum_{i=1}^n p_ie^{t_i}\right]^m \\
\mathrm{Cov}(X_i, X_j) &= \frac{\partial^2}{\partial t_i \partial t_j} \log M_{\boldsymbol{X}}(\boldsymbol{t})\Big|_{\boldsymbol{t}=\boldsymbol{0}} \\
&= \frac{\partial}{\partial t_j} m \frac{p_ie^{t_i}}{\sum p_ke^{t_k}} \\
&= -\frac{mp_ie^{t_i}p_je^{t_j}}{\sum t_ke^{t_k}}\Big|_{\boldsymbol{t}=\boldsymbol{0}} \\
&= -mp_ip_j
\end{aligned}$$

4.40

(a)

Let $W = \frac{X}{1-Y}$, then integrating the bivariate density function is

$$\begin{aligned}
1 &= C\int_0^1\int_0^{1-y} x^{a-1}y^{b-1}(1-x-y)^{c-1}dxdy \\
&= C\int_0^1\int_0^1 (w(1-y))^{a-1}y^{b-1}[1-w(1-y)-y]^{c-1}(1-y)dwdy \\
&= C\int_0^1\int_0^1 y^{b-1}(1-y)^{a+c-1}w^{a-1}(1-w)^{c-1}dwdy \\
&= C\frac{\Gamma(b)\Gamma(a+c)}{\Gamma(a+b+c)}\frac{\Gamma(a)\Gamma(c)}{\Gamma(a+c)} \\
C &= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}
\end{aligned}$$

(b)

(i) Let $W = \frac{Y}{1-X}$, the marginal density of X is

$$\begin{aligned}
f_X(x) &= \int_0^{1-x}\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}x^{a-1}y^{b-1}(1-x-y)^{c-1}dy \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}x^{a-1}(1-x)^{b+c-1}\int_0^1 w^{b-1}(1-w)^{c-1}dw \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}x^{a-1}(1-x)^{b+c-1}\frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)}x^{a-1}(1-x)^{b+c-1}
\end{aligned}$$

Thus, $X \sim$beta$(a, b+c)$

(ii) Let $T = \frac{X}{1-Y}$, the marginal density of Y is

$$\begin{aligned}
f_Y(y) &= \int_0^{1-y}\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}x^{a-1}y^{b-1}(1-x-y)^{c-1}dx \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}y^{b-1}(1-y)^{a+c-1}\int_0^1 t^{a-1}(1-t)^{c-1}dt \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}y^{b-1}(1-y)^{a+c-1}\frac{\Gamma(a)\Gamma(c)}{\Gamma(a+c)} \\
&= \frac{\Gamma(a+b+c)}{\Gamma(b)\Gamma(a+c)}y^{b-1}(1-y)^{a+c-1}
\end{aligned}$$

Thus, $Y \sim$beta$(b, a+c)$

(c)

(i) The conditional distribution of $Y|X = x$ is

$$\begin{aligned}
f(y|x) = \frac{f(x,y)}{f(x)} &= \frac{\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}x^{a-1}y^{b-1}(1-x-y)^{c-1}}{\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)}x^{a-1}(1-x)^{b+c-1}} \\
&= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)}\left(\frac{y}{1-x}\right)^{b-1}\left(1-\frac{y}{1-x}\right)^{c-1}\frac{1}{1-x}
\end{aligned}$$

(ii) Consider a bivariate transformation

$$\begin{matrix} U = \frac{Y}{1-X} \\ V = 1-X \end{matrix} \Rightarrow \left\{\begin{matrix} X = 1-V \\ Y = UV \end{matrix}\right. \Rightarrow |J| = \begin{vmatrix} 0 & -1 \\ V & U \end{vmatrix} = |V|$$

The joint density function of (U, V) is

$$\begin{aligned} f(u,v) &= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}(1-v)^{a-1}(uv)^{b-1}[1-(1-v)-uv]^{c-1}v \\ &= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}v^{c+b-1}(1-v)^{a-1}u^{b-1}(1-u)^{c-1} \end{aligned}$$

Thus, U, V are independent beta random variables, and $U = \frac{Y}{1-X}$ ~beta(b,c)
(d)

$$\begin{aligned} EXY &= \int_0^1 \int_0^{1-y} xy \cdot \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} x^{a-1} y^{b-1} (1-x-y)^{c-1} dxdy \\ &= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \int_0^1 \int_0^1 w^a (1-w)^{c-1} y^b (1-y)^{a+c} dwdy \\ &= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \frac{\Gamma(a+1)\Gamma(c)}{\Gamma(a+1+c)} \frac{\Gamma(b+1)\Gamma(a+c+1)}{\Gamma(b+1+a+c+1)} \\ &= \frac{\Gamma(a+b+c)}{\Gamma(a+b+c+2)} \frac{\Gamma(a+1)}{\Gamma(a)} \frac{\Gamma(b+1)}{\Gamma(b)} \\ &= \frac{ab}{(a+b+c+1)(a+b+c)} \end{aligned}$$

$$\begin{aligned} \text{Cov}(X,Y) &= EXY - EXEY \\ &= \frac{ab}{(a+b+c+1)(a+b+c)} - \frac{\Gamma(a)}{\Gamma(a+b+c)} \frac{\Gamma(b)}{\Gamma(a+b+c)} \end{aligned}$$

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$$\begin{aligned} EXY \cdot Y &= EXY^2 = EX \cdot EY^2 = \mu_X \cdot (\mu_Y^2 + \sigma_Y^2) \\ \text{Cov}(XY, Y) &= EXY \cdot Y - EXY \cdot EY \\ &= \mu_X \mu_Y^2 + \mu_X \sigma_Y^2 - \mu_X \mu_Y \cdot \mu_Y = \mu_X \sigma_Y^2 \\ \text{Var}(XY) &= EX^2Y^2 - (EXY)^2 = (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2 \mu_Y^2 \\ \text{Cor}(XY, Y) &= \frac{\mu_X \sigma_Y^2}{\sqrt{\sigma_X^2 \sigma_Y^2 + \mu_X^2 \sigma_X^2 + \mu_Y^2 \sigma_X^2} \sqrt{\sigma_Y^2}} \end{aligned}$$

4.45
(a) Denote $c_1 = (2\pi\sqrt{(1-\rho^2)\sigma_X^2\sigma_Y^2})^{-1}$

$$\begin{aligned} f_X(x) &= c_1 \cdot e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \int_{\mathbb{R}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right)\right] dy \\ &= c_1 \cdot e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \int_{\mathbb{R}} \exp\left[-\frac{1}{2(1-\rho^2)}(z^2 - 2\rho c_2 z)\right] \sigma_Y dz \qquad \text{Let } z = \frac{y-\mu_Y}{\sigma_Y}, \quad c_2 = \frac{x-\mu_X}{\sigma_X} \\ &= c_1 \cdot e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \sigma_Y \int_{\mathbb{R}} \exp\left[-\frac{(z-\rho c_2)^2}{2(1-\rho^2)}\right] \exp\left[-\frac{-(\rho c_2)^2}{2(1-\rho^2)}\right] dz \\ &= \frac{1}{2\pi\sqrt{(1-\rho^2)\sigma_X^2\sigma_Y^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \sigma_Y \exp\left[\frac{\rho^2((x-\mu_X)/\sigma_X)^2}{2(1-\rho^2)}\right] \sqrt{2\pi(1-\rho^2)} \\ &= \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left[\frac{-\left(\frac{x-\mu_X}{\sigma_X}\right)^2(1-\rho^2)}{2(1-\rho^2)}\right] \\ &= \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right] \end{aligned}$$

Thus, the marginal distribution of X is $\mathrm{n}(\mu_X, \sigma_X^2)$.
By using the similar technique, we can derive the marginal distribution of Y accordingly, which will be $\mathrm{n}(\mu_Y, \sigma_Y^2)$
(b) Denote $c_3 = \frac{y-\mu_Y}{\sigma_Y}$

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{\frac{1}{2\pi\sqrt{1-\rho^2}\sigma_X\sigma_Y}\exp\left[-\frac{1}{2(1-\rho^2)}(c_2^2-2\rho c_2c_3+c_3^2)\right]}{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}c_2^2}} \\
&= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\sigma_Y}\exp\left[-\frac{1}{2(1-\rho^2)}\left(c_2^2-2\rho c_2c_3+c_3^2\right)+\frac{(1-\rho^2)}{2(1-\rho^2)}c_2^2\right] \\
&= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\sigma_Y}\exp\left[-\frac{1}{2(1-\rho^2)}\left(c_3-\rho c_2\right)^2\right] \\
&= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\sigma_Y}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{y-\mu_Y}{\sigma_Y}-\rho\frac{x-\mu_X}{\sigma_X}\right)^2\right] \\
&= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\sigma_Y}\exp\left[-\frac{1}{2(1-\rho^2)\sigma_Y^2}\left((y-\mu_Y)-\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right)^2\right] \\
&= \frac{1}{\sqrt{2\pi}\sqrt{(1-\rho^2)\sigma_Y^2}}\exp\left[-\frac{1}{2(1-\rho^2)\sigma_Y^2}\left(y-[\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)]\right)^2\right]
\end{aligned}$$

Thus, the conditional distribution of Y given X is $\mathrm{n}(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X), \sigma_Y^2(1-\rho^2))$
(c)
Let $W = aX + bY$. The MGF of W is given by

$$\begin{aligned}
M_W(t) &= E(e^{t(aX+bY)}) \\
&= M_{X,Y}(at, bt) \\
&= \exp\left[(\mu_X \;\; \mu_Y)\begin{pmatrix} at \\ bt \end{pmatrix}\right]\cdot\exp\left[(at \;\; bt)\begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}\begin{pmatrix} at \\ bt \end{pmatrix}\right] \\
&= e^{(a\mu_X+b\mu_Y)t}\cdot\exp\left[(a^2\sigma_X^2+2ab\rho\sigma_X\sigma_Y+b^2\sigma_Y^2)t^2\right]
\end{aligned}$$

By uniqueness of MGF, $aX + bY \sim \mathrm{n}(a\mu_X + b\mu_Y, a^2\sigma_X^2 + 2ab\rho\sigma_X\sigma_Y + b^2\sigma_Y^2)$

4.46
(a)

$$\begin{aligned}
EX &= E(a_XZ_1+b_XZ_2+c_X) = a_XE(Z_1)+b_XEZ_2+c_X = c_X \\
EY &= E(a_YZ_1+b_YZ_2+c_Y) = a_YE(Z_1)+b_YEZ_2+c_Y = c_Y \\
\mathrm{Var}X &= \mathrm{Var}a_XZ_1+b_XZ_2 = a_X^2+b_X^2 \\
\mathrm{Var}Y &= \mathrm{Var}a_YZ_1+b_YZ_2 = a_Y^2+b_Y^2 \\
\mathrm{Cov}(X,Y) &= \mathrm{Cov}(a_XZ_1+b_XZ_2+c_X, a_YZ_1+b_YZ_2+c_Y) \\
&= \mathrm{Cov}(a_XZ_1, a_YZ_1)+\mathrm{Cov}(b_XZ_2, b_YZ_2) \\
&= a_Xa_Y+b_Xb_Y
\end{aligned}$$

(b) By plugging a_X, b_X, c_X, a_Y, b_Y and c_Y into results in (a), we have

$$\begin{aligned}
EX &= c_X = \mu_X \\
EY &= c_Y = \mu_Y \\
\mathrm{Var}X &= a_X^2+b_X^2 = \frac{1+\rho}{2}\sigma_X^2+\frac{1-\rho}{2}\sigma_X^2 = \sigma_X^2 \\
\mathrm{Var}Y &= a_Y^2+b_Y^2 = \frac{1+\rho}{2}\sigma_Y^2+\frac{1-\rho}{2}\sigma_Y^2 = \sigma_Y^2 \\
\rho_{XY} &= \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}X\cdot\mathrm{Var}Y}} \\
&= \frac{\frac{1+\rho}{2}\sigma_X\sigma_Y-\frac{1-\rho}{2}\sigma_X\sigma_Y}{\sqrt{\sigma_X^2\sigma_Y^2}} \\
&= \frac{\rho\sigma_X\sigma_Y}{\sigma_X\sigma_Y} = \rho
\end{aligned}$$

(c) Denote $\boldsymbol{c}_1 = (a_X \;\; b_X)',\; \boldsymbol{c}_2 = (a_Y \;\; b_Y)', \boldsymbol{a} = (a_X \;\; a_Y)', \boldsymbol{b} = (b_X \;\; b_Y)', \boldsymbol{Z} = (Z_1 \;\; Z_2)'$
The joint MGF of X, Y is

$$\begin{aligned}
M_{X,Y}(t_1,t_2) &= E(\exp\left[t_1(\boldsymbol{c}_1'\boldsymbol{Z}+c_X), t_2(\boldsymbol{c}_2'\boldsymbol{Z}+c_Y)\right]) \\
&= E(\exp\left[t_1(\boldsymbol{c}_1'\boldsymbol{Z}), t_2(\boldsymbol{c}_2'\boldsymbol{Z})\right])e^{t_1c_X+t_2c_Y} \\
&= e^{\boldsymbol{\mu}'\boldsymbol{t}} M_{Z_1,Z_2}(\boldsymbol{a}'\boldsymbol{t}, \boldsymbol{b}'\boldsymbol{t}) \\
&= e^{\boldsymbol{\mu}'\boldsymbol{t}} \exp\left[\frac{1}{2}\boldsymbol{t}'\begin{pmatrix} \boldsymbol{a}' \\ \boldsymbol{b}' \end{pmatrix} \boldsymbol{I}(\boldsymbol{a} \;\; \boldsymbol{b})\boldsymbol{t}\right] \\
&= e^{\boldsymbol{\mu}'\boldsymbol{t}} \exp\left[\frac{1}{2}\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t}\right]
\end{aligned}$$

where $\boldsymbol{\mu} = (\mu_X \;\; \mu_Y)', \boldsymbol{t} = (t_1 \;\; t_2)'$ and $\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{a}' \\ \boldsymbol{b}' \end{pmatrix}(\boldsymbol{a} \;\; \boldsymbol{b}) = \begin{bmatrix} \boldsymbol{a}'\boldsymbol{a} & \boldsymbol{a}'\boldsymbol{b} \\ \boldsymbol{b}'\boldsymbol{a} & \boldsymbol{b}'\boldsymbol{b} \end{bmatrix} = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$
Thus, by uniqueness of MGF, $(X,Y) \sim \mathrm{n}_2\left(\boldsymbol{\mu} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}\right)$
(d)
Let

$$\begin{aligned}
c_X^* &= \mu_X, \quad c_Y^* = \mu_Y \\
a_X^* &= \sqrt{\frac{1+\rho}{2}}\sigma_X, \quad b_X^* = -\sqrt{\frac{1-\rho}{2}}\sigma_X \\
a_Y^* &= \sqrt{\frac{1+\rho}{2}}\sigma_Y, \quad b_Y^* = \sqrt{\frac{1-\rho}{2}}\sigma_Y
\end{aligned}$$

Then, $a_X^*, b_X^*, a_Y^*, b_Y^*, c_X^*$ and c_Y^* is another solution for the system.
There are infinite solutions for the system since there are six equations with only five independent variable.

4.53
The equation Ax^2+Bx+c has real root if and only if the discriminant $B^2 \geq 4AC$, or equivalently $-2\log B \leq -\log 4 - \log A - \log C$
Let $X = -2\log B \sim$exponential(2), $Y = -\log A - \log C \sim \Gamma(2,1)$, the probability of having real root is

$$\begin{aligned}
P(X \leq -\log 4 + Y) &= P(\log 4 \leq Y - X) \\
&= \int_{\log 4}^{\infty}\int_0^{y-\log 4} \frac{1}{2}e^{-\frac{x}{2}} y e^{-y} dxdy \\
&= \int_{\log 4}^{\infty} (1-e^{-\frac{y-\log 4}{2}}) y e^{-y} dy \\
&= \int_{\log 4}^{\infty} y e^{-y} dy - e^{\frac{\log 4}{2}} \int_{\log 4}^{\infty} y e^{-\frac{3}{2}y} dy \\
&= \left(\frac{\log 2}{2} + \frac{1}{4}\right) - 2\left(\frac{\log 2}{6} + \frac{1}{18}\right) \\
&= \frac{\log 2}{6} + \frac{5}{36}
\end{aligned}$$

4.54
We first find the density function for $T_2 = \sum_{i=1}^n -\log X_i$, then let $T = \prod_{i=1}^n X_i = e^{-T_2}$.
Since $X_i \sim \mathrm{U}(0,1)$ independently, we have $-\log X_i \sim$Gamma(1,1), independently. Therefore, by reproductiveness of Gamma distribution, $T_2 = \sum -\log X_i \sim$Gamma(n,1).
Consider the transformation

$$T = \phi(T_2) = e^{-T_2}, \; T_2 = \phi^{-1}(T) = -\log T, \; |J| = |1/T|$$

The density function of T is given by

$$\begin{aligned} f(t) &= \frac{1}{\Gamma(n)}(-\log t)^{n-1}e^{\log t}\frac{1}{t} \\ &= \frac{1}{\Gamma(n)}(-\log t)^{n-1}e^{\log t}\frac{1}{t} \\ &= \frac{1}{\Gamma(n)}(-\log t)^{n-1}, \quad t \in (0,1) \end{aligned}$$

4.63
Since $g(X) = \log(X)$ is a concave function, by Jensen's inequality

$$\begin{aligned} 0 = EZ = Eg(X) &< g(EX) = \log EX \quad \text{take exponential on both side} \\ e^0 &< EX \end{aligned}$$

Thus, EX is greater than 1