Practice Midterm Exam, Fall 2008, Statistics 521

Note: You must clearly explain all the steps.

1. Prove that if $P(A) > 0$ and $P(B) > 0$, then
(i) If A and B are mutually exclusive, they cannot be independent.
(ii) If A and B are independent, they cannot be mutually exclusive. [20]

2. If the sample space $\mathcal{S} = A_1 \cup A_2$ and if $P(A_1) = 0.8$ and $P(A_2) = 0.5$, find $P(A_1 \cap A_2)$.

3. Two people each toss a fair coin n times. Find the probability that they will toss the same number of heads. [10]

4. In how many ways can 4 different pair of shoes be laid in a row if
(i) there are no restrictions on the laying arrangement. [5]
(ii) all the four left-foot shoes are laid together and all the four right-foot shoes are also laid together. [5]
(iii) at least one pair of shoes are not laid next to each other. [5]

5. In answering a question on a multiple-choice test, a student either (correctly) knows the answer or guesses. Let p be the probability that the student knows the answer and $1-p$ be the probability that the student guesses. Assume that a student who guesses at the answer guesses completely at random, that is the student will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer, given that he or she answered it correctly? [20]

6. Let the pdf $f(x)$ be positive on and only on the non-negative integers. Suppose that

$$f(x) = \frac{4}{x} f(x-1), \quad \text{for } x = 1, 2, \cdots$$

Find $f(x)$. [15]

7. Let X be a continuous random variable with the density function

$$f_X(x) = \begin{cases} \frac{1}{3}x^2, & \text{if} \quad -2 \leq x \leq 1 \\ 0. & \text{o.w.} \end{cases}$$

Define $Y = X^2$. Find the density function of Y. [20]

8. The random variable X have the density

$$f_X(x) = xe^{-x}, \qquad x > 0.$$

(i) Compute $E(X)$ and $Var(X)$.
(ii) Denote $\mu = E(X)$ and $\sigma^2 = Var(X)$. Compute $P(|X - \mu| \geq \sqrt{2}\sigma)$.
(iii) Find the density of $Y = X^3$, and compute the expected value $E(Y)$? [20]

9. Let X be a discrete random variable taking values $0, 1, 2, \ldots$ with probabilities $P(X = 0) = 1 - \alpha + \alpha p$ and for $k = 1, 2, \ldots$, $P(X = k) = \alpha p q^k$, where $q = 1 - p$, $0 < p < 1$, $0 < \alpha < 1$.
(i) Find the moment generating function of X.
(ii) Compute the $E(X)$ and $Var(X)$ based on the moment generating function obtained from (i). [20]

10. Assume that the moment-generating function of a random variable X is

$$M_X(t) = (\frac{3}{5} + \frac{2}{5}e^t)^6.$$

(i) What is the distribution of X?
(ii) Compute the $E(X)$ and $Var(X)$.

11. Let X have the log-normal distribution having the density

$$f(x) = \frac{1}{\sqrt{2\pi}x} e^{-(\log x)^2/2}, \quad 0 < x < \infty.$$

Find the density function of $Y = \sqrt{X}$.

12. Let X have the standard Cauchy density function

$$f(x) = \frac{1}{\pi(1 + x^2)}, \quad -\infty < x < \infty.$$

Find the density function of $Y = 1/X$ and the median of $Z = Y^2 = 1/X^2$.