Show ALL your work, along with JUSTIFICATION for the steps you take.

1. The following multiple choice questions will receive no partial credit, so no justification is required.

   (a) (5 points) Let $Y_1, Y_2, Y_3, Y_4, Y_5$ be a random sample from a normal distribution with unknown mean $\mu$ and unknown variance $\sigma^2$. Which of the following can appropriately be used as a $(1 - \alpha)100\%$ confidence interval for $\mu$?

   I. $\left( \bar{Y} - z_{\alpha/2} \frac{\sigma}{\sqrt{5}}, \bar{Y} + z_{\alpha/2} \frac{\sigma}{\sqrt{5}} \right)$
   II. $\left( \bar{Y} - z_{\alpha/4} \frac{\sigma}{\sqrt{5}}, \bar{Y} + z_{3\alpha/4} \frac{\sigma}{\sqrt{5}} \right)$
   III. $\left( \bar{Y} - z_{\alpha/2} \frac{S}{\sqrt{5}}, \bar{Y} + z_{\alpha/2} \frac{S}{\sqrt{5}} \right)$

   A. I only       B. II only      C. III only      D. I and II only
   E. None of the above

   (b) (5 points) You are given two confidence intervals, $5 \pm 4$ and $(5 - 3, 5 + 5)$, for the mean $\mu$ of a population, and told that they have the same confidence coefficient. Which of the following is true?

   A. The confidence interval $5 \pm 4$ is better.
   B. The confidence interval $(5 - 3, 5 + 5)$ is better.
   C. The confidence intervals are equally desirable.
   D. There is insufficient information on which to base a comparison.

   (c) (5 points) A random sample yields $(1.85, 9.21)$ as a $99\%$ confidence interval for $\mu$. Which of the following statements are true?

   I. The probability is 0.99 that $\mu$ falls between 1.85 and 9.21.
   II. Ninety nine percent of the population values will fall between 1.85 and 9.21.
   III. If another observation is collected, there is a $99\%$ chance that new observation will be greater than 1.85, but less than 9.21.

   A. I only       B. II only      C. III only      D. I and III only
   E. None of the above
2. (42 points) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a population with p.d.f.

$$f(y) = \begin{cases} \frac{y}{\theta} e^{-y^2/2\theta} & y > 0 \\ 0 & \text{otherwise}. \end{cases}$$

Furthermore, this p.d.f. can be used to show that $E(Y) = \frac{1}{2} \sqrt{\pi \sqrt{2\theta}}$, $E(Y^2) = 2\theta$, and $E(Y^4) = 8\theta^2$. (Do not show these!)

(a) Find the Method of Moments Estimator of $\theta$.
(b) Find the Maximum Likelihood Estimator of $\theta$.
(c) Is the estimator in (a) consistent for estimating $\theta$? You may use, without proof, the fact that $\text{Var}(\bar{Y}^2)$ is approximately equal to $[4\pi \theta^2 (1 - \pi/4)]/n$.
(d) Is the estimator in (b) consistent for estimating $\theta$?
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3. (28 points) Suppose \( Y_{11}, Y_{12}, \ldots, Y_{1n_1} \) represents a random sample from a normal distribution with mean \( \mu_1 \) and variance \( \sigma_1^2 \). An independent random sample \( Y_{21}, Y_{22}, \ldots, Y_{2n_2} \) also has a normal distribution but with mean \( \mu_2 \) and variance \( \sigma_2^2 \). We would like to find a \((1 - \alpha)100\%\) confidence interval for \( \frac{\sigma_1^2}{\sigma_2^2} \).

(a) Show that

\[
\frac{S_1^2}{\sigma_1^2} \cdot \frac{\sigma_2^2}{S_2^2}
\]

is a pivotal random variable for \( \frac{\sigma_1^2}{\sigma_2^2} \). What is its distribution (just quote a result from class)?

(b) Use the pivotal random variable given in (a) to complete the steps in finding the desired confidence interval.

(c) Suppose data has been collected and a 95% confidence interval for \( \frac{\sigma_1^2}{\sigma_2^2} \) is obtained as \((0.289, 3.734)\). Does this mean the probability is 0.95 that \( \frac{\sigma_1^2}{\sigma_2^2} \) is at least 0.289 and at most 3.734? Answer yes or no. No justification is required.
4. (15 points) The XYZ company has installed a safety device to prevent toxic waste from entering a nearby river. This device has a limited lifetime. If the company replaces the device earlier than needed, the cost is about $20,000. On the other hand, if the company doesn’t replace the device in time, so that there is discharge of toxic waste into the river, it could cause death of nearby residents, clean up costs for the company, and possible law suits totalling hundreds of millions of dollars. The company would like to estimate the mean lifetime $\theta$ of the device. Comment on whether they should try to find an unbiased estimator of $\theta$, a negatively biased estimator \( [i.e., E(\hat{\theta}) < \theta] \), or a positively biased estimator \( [i.e., E(\hat{\theta}) > \theta] \)? Give reasons for the recommendation you make.
Bonus (5 points): Attempt the following only after you are satisfied with all other responses. Partial credit will not be given.

Refer to question 2. Show that \( E(Y) = \frac{1}{2} \sqrt{\frac{\pi}{\theta}}. \)