1. (30 points) Suppose $Y_1, Y_2, \ldots, Y_n$ forms a random sample from a $Beta(1, \beta)$ distribution.

(a) Find $\hat{\beta}_{\text{mom}}$, the method of moments estimator of $\beta$.

(b) Find $\hat{\beta}_{\text{mle}}$, the maximum likelihood estimator of $\beta$.

(c) A sample of size $n = 20$ gives a sample mean of 0.1 and $\sum_{i=1}^{20} \ln(1 - y_i) = -2$. What are the sample method of moments and maximum likelihood estimates of $\beta$?
2. (40 points) Suppose $Y_1, Y_2, \ldots, Y_n$ forms a random sample from a $Bin(1, p)$ distribution. Consider the following estimators of $p$: $\hat{p}_1 = Y_1; \hat{p}_2 = \frac{1}{n}(Y_1 + Y_2 + \cdots + Y_{n-1}); \hat{p}_3 = \frac{1}{n}(Y_4 + Y_5 + \cdots + Y_n)$.

(a) Which, if any, of these three estimators are unbiased?

(b) Which, if any, of these three estimators are consistent?

(c) Using the results of (a) and (b), modify the biased but consistent estimators to make them unbiased.

(d) Can you find a minimum variance unbiased estimator (MVUE) of $p$?
3. (30 points) Suppose $Y_1, Y_2, \ldots, Y_n$ forms a random sample from the exponential distribution with mean $\beta$.

(a) Show that a $(1 - \alpha)100\%$ confidence interval for $\beta$ is

$$
\left( \frac{2 \sum_{i=1}^{n} Y_i}{\chi^2_{2n,\alpha/2}}, \frac{2 \sum_{i=1}^{n} Y_i}{\chi^2_{2n,1-\alpha/2}} \right),
$$

where $\chi^2_{2n,\alpha/2}$ and $\chi^2_{2n,1-\alpha/2}$ are such that $P(\chi^2_{2n} > \chi^2_{2n,\alpha/2}) = \alpha/2$ and $P(\chi^2_{2n} > \chi^2_{2n,1-\alpha/2}) = 1 - \alpha/2$.

**Hint:** You may use the fact that $\frac{2 \sum_{i=1}^{n} Y_i}{\beta} \sim \chi^2_{2n}$. *Do not prove this.*

(b) It is known that the lifetimes of bulbs manufactured by company XYZ follows an exponential distribution. The company claims that the mean lifetime of their bulbs is 900 hours. A random sample of 20 bulbs yields an average lifetime of 750 hours. Using a 95% confidence interval of the form in (a), do you find support for the company’s claim? Explain.
Bonus (5 points): Attempt the following only after you are satisfied with all other responses. Partial credit will not be given.

A random sample aimed at estimating the mean concentration of toxin in Lake Johnson yields a 95% confidence interval of (1 ppm, 10 ppm) for the mean concentration. Therefore, we know that the probability is 0.95 that the true mean concentration is somewhere between 1 and 10 ppm.

Do you agree or disagree? No justification required.