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Complete Convergence of Martingale Arrays

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We study complete convergence of martingale arrays under rather weak conditions. Our results considerably strengthen many of the results available in the literature. As a tool, we establish a martingale analogue of an inequality of Hoffman-Jørgensen which was earlier known only for independent random variables.

KEY WORDS: Complete convergence; Hoffman-Jørgensen inequality; martingale.

1. INTRODUCTION

Let $\{(X_{nk}, \mathscr{F}_{nk}): k \geqslant 1, n \geqslant 1\}$ be a triangular array of martingale differences with finite moments of appropriate order. Modifying certain ideas of Li *et al.*,[8] we show that under certain natural conditions, $\sum_{k=1}^{\infty} X_{nk}$ converges completely to zero, i.e., for any $\varepsilon > 0$, $\sum_{n=1}^{\infty} P\{|\sum_{k=1}^{\infty} X_{nk}| > \varepsilon\} < \infty$. This result is then applied to certain special arrays to obtain important extensions of some of the existing results on complete convergence.

The notion of complete convergence was introduced by Hsu and Robbins.[6] The importance of complete convergence arises from the fact that it gives additional information on the rates of convergence in an almost sure convergence. For a discussion on complete convergence, more references and a general review, Gut[3] may be consulted. Our results extend considerably the results of Hsu and Robbins,[6] Chow,[2] Yu[11] and several of the results of Li *et al.*[8] We establish a new martingale inequality

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(see our Theorem 1), which is a variation of an inequality due to Hoffman-Jørgensen[5] for independent random variables. This inequality is then used as the principal tool for establishing the results on complete convergence. With the help of this inequality, we are able to extend many results previously known for sum of independent random variables. Since we consider variables which are neither independent nor identically distributed, no attempt has been made to show the necessity of the stated conditions. Study of complete convergence in the dependent setup has begun only recently. Yu[11] considered the case of martingale differences and Shao[9, 10] studied complete convergence under mixing conditions; see also the references therein. Complete convergence for moving average process has been studied in Li *et al.*[7] Like Yu,[11] we also consider martingale difference sequences. However, our results are much more general than Yu's,[11] and moreover, the techniques used are completely different.

2. MAIN RESULTS

In this section, we present all the main results. The proofs are deferred to Section 3.

Our first result is a martingale analogue of an important inequality due to Hoffman-Jørgensen,[5] which is elegantly used in the proof of complete convergence for independent random variables [see Li *et al.*[8]]. A result somewhat similar to our Theorem 1, part (a), called "the good λ inequality," was earlier obtained by Burkholder[1] [Eq. (12.3)]. A stronger form of Burkholder's result, with exponential rather than quadratic decay of the constant on the right-hand side of the inequality, was obtained by Hitczenko[4] [Prop. 3.2]. However, both results need the assumption that the martingale-differences are dominated by a predictable process which we do not assume. Moreover, it turns out that the particular form presented here suits better for the purpose of establishing the results on complete convergence. Finally, by repeated applications of part (a) of Theorem 1, we can obtain decay rate faster than any given power in part (b).

Theorem 1. Let $\{(X_k, \mathscr{F}_k), k \geqslant 1\}$ be square-integrable martingale differences such that for some constant M, $\sum_{k=1}^{\infty} E(X_k^2 \mid \mathscr{F}_{k-1}) \leqslant M$ a.s., where $\mathscr{F}_0$ is a trivial σ-field.. Let $S_k = X_1 + \cdots + X_k$, $k \geqslant 1$. Then

(a) for $t, r, s > 0$,

$$P\{\sup_{k \geqslant 1} |S_k| \geqslant t + r + s\} \leqslant P\{\sup_{k \geqslant 1} |X_k| \geqslant s\} + \frac{M}{r^2} P\{\sup_{k \geqslant 1} |S_k| \geqslant t\} \tag{2.1}$$

and

(b) for $t > 0$ and an integer $m \geqslant 1$,

$$P\{\sup_{k\geqslant 1} |S_k| \geqslant 2mt\} \leqslant \frac{1-(M/t^2)^{m-1}}{1-(M/t^2)} P\{\sup_{k\geqslant 1} |X_k| \geqslant t\} + \frac{1}{t}\left(\frac{M}{t^2}\right)^m \tag{2.2}$$

The next result is on complete convergence of square-integrable martingale-arrays and is an L^2-martingale analogue of Theorem 1 of Li *et al.*[8]

Theorem 2. Let $\{(X_{nk}, \mathscr{F}_{nk}), k \geqslant 1\}$ be sequences of square-integrable martingale differences. Suppose that there exist constants $\{M_n\}$ such that $\sum_{k=1}^{\infty} E(X_{nk}^2 \mid \mathscr{F}_{n,k-1}) \leqslant M_n$ a.s., where $\mathscr{F}_{n0}$ is trivial for all n. Let $\{c_n\}$ be a sequence of nonnegative numbers satisfying $\sum_{n=1}^{\infty} c_n M_n^{\lambda} < \infty$ for some $\lambda > 0$ and

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{\infty} P\{|X_{nk}| > \varepsilon\} < \infty \qquad \forall \varepsilon > 0 \tag{2.3}$$

Then

$$\sum_{n=1}^{\infty} c_n P\left\{\sup_{k\geqslant 1}\left|\sum_{i=1}^{k} X_{ni}\right| > \varepsilon\right\} < \in \infty \qquad \forall \varepsilon > 0 \tag{2.4}$$

The following couple of examples show that Theorem 1 is actually applicable to martingales distinct from the sequence of partial sums of mean-zero independent random variables also. The same remark applies to the other results in this paper also.

Example 1. Let $\{Y_{nk}\}$ be an array of row-wise independent mean-zero random variables satisfying $|Y_{nk}| \leqslant c_{nk}$ a.s. for some positive constant satisfying $\sup_{n\geqslant 1} \sum_{k=1}^{\infty} c_{nk} \leqslant K < \infty$ and $\sum_{k=1}^{\infty} EY_{nk}^2 = O(n^{-\alpha})$ for some $\alpha > 0$. Consider the martingale $M_{nk} = \prod_{j=1}^{k} (1 + Y_{nj})$ and the corresponding difference sequence $X_{nk} = M_{nk} - M_{n,k-1}$ with the natural filtration. Then $E(X_{nk}^2 \mid \mathscr{F}_{n,k-1}) = \prod_{j=1}^{k-1} (1 + Y_{nj})^2 \, EY_{nk}^2$ and so

$$\sum_{k=1}^{\infty} E(X_{nk}^2 \mid \mathscr{F}_{n,k-1}) \leqslant \prod_{j=1}^{\infty} (1 + c_{nj})^2 \sum_{k=1}^{\infty} EY_{nk}^2 = O(n^{-\alpha})$$

Hence by Theorem 2, the sequence $\prod_{k=1}^{\infty} (1 + Y_{nk})$ converges completely to 1.

Example 2. Let $\{Y_{nk}\}$ be an array of row-wise independent positive random variables. Let $t > 0$ and $\psi_{nk}(t) = E(\exp[-tX_{nk}])$. Consider the martingale $M_{nk} = \exp(-t\sum_{j=1}^{k} Y_{nj})/\prod_{j=1}^{k} \psi_{nj}(t)$ and the corresponding differences $X_{nk} = M_{nk} - M_{n,k-1}$. Assume that $\inf_{n \geqslant 1} \prod_{k=1}^{\infty} \psi_{nk}(t) > 0$ and $\sum_{k=1}^{\infty} (\psi_{nk}(2t)/(\psi_{nk}(t))^2 - 1) = O(n^{-\alpha})$ for some $\alpha > 0$. (For example, Poisson variables with parameters λ_{nk} satisfying $\sum_{k=1}^{\infty} \lambda_{nk} = O(n^{-\alpha})$ meet the condition.) Then the array $\{X_{nk}\}$ satisfies the hypothesis of Theorem 2 and so $\exp[-t\sum_{k=1}^{\infty} Y_{nk}]/\prod_{k=1}^{\infty} \psi_{nk}(t)$ converges completely to 1.

We now consider complete convergence for L^p-martingales. Theorem 3 and Corollaries 1 and 2 have been formulated by the referee. Theorem 3 allows us to unify and extend the results we originally obtained as martingale analogues of (the direct halves of) Theorems 2–4 of Li *et al.*[8]

Theorem 3. Let $\{(X_{nk}, \mathscr{F}_{nk}), k \geqslant 1\}$ be sequences of square-integrable martingale differences satisfying $\sup_{n,k} E(X_{nk}^2 \mid \mathscr{F}_{n,k-1}) \leqslant K$ a.s. Let $\{b_{nk}\}$ be an array of real numbers such that $\sum_{k=1}^{\infty} b_{nk}^2 = O(n^{-\alpha})$ for some $\alpha > 0$. Let U_k, V_n and W be random variables satisfying

$$\sup_{n \geqslant 1} P\{|X_{nk}| \geqslant x\} \leqslant CP\{|U_k| \geqslant x\} \qquad \forall x > 0, \quad \forall k \geqslant 1 \tag{2.5}$$

$$\sup_{k \geqslant 1} P\{|X_{nk}| \geqslant x\} \leqslant CP\{|V_n| \geqslant x\} \qquad \forall x > 0, \quad \forall n \geqslant 1 \tag{2.6}$$

$$\sup_{n,k \geqslant 1} P\{|X_{nk}| \geqslant x\} \leqslant CP\{|W| \geqslant x\} \qquad \forall x > 0 \tag{2.7}$$

for some finite constant C. Let $\xi : [0, \infty) \to [0, \infty)$ be a function such that $\xi(x) \geqslant 1$ for all $x > 1$. Set

$$\xi_{\bullet k}(x) = \sum_{n=1}^{\infty} \xi(|b_{nk}x|), \qquad \xi_{n\bullet}(x) = \sum_{k=1}^{\infty} \xi(|b_{nk}x|)$$
$$\xi_{\bullet\bullet}(x) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \xi(|b_{nk}x|) \tag{2.8}$$

Then any one of the following four conditions

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} E\xi\left(\frac{|b_{nk}X_{nk}|}{\varepsilon}\right) < \infty \qquad \forall \varepsilon > 0 \tag{2.9}$$

$$\sum_{k=1}^{\infty} E\xi_{\bullet k}\left(\frac{|U_k|}{\varepsilon}\right) < \infty \qquad \forall \varepsilon > 0 \tag{2.10}$$

$$\sum_{n=1}^{\infty} E\xi_{n\bullet}\left(\frac{|V_n|}{\varepsilon}\right) < \infty \qquad \forall \varepsilon > 0 \tag{2.11}$$

$$E\xi_{\bullet\bullet}\left(\frac{|W|}{\varepsilon}\right) < \quad \infty \qquad \forall \varepsilon > 0 \tag{2.12}$$

implies

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} P\{|b_{nk} X_{nk}| > \varepsilon\} < \infty \qquad \forall \varepsilon > 0 \tag{2.13}$$

and hence

$$\sum_{n=1}^{\infty} P\left\{\sup_{k \geqslant 1} \left| \sum_{i=1}^{k} b_{ni} X_{ni} \right| > \varepsilon\right\} < \infty \qquad \forall \varepsilon > 0 \tag{2.14}$$

Remark 1. Set $R(x) = \sup_{n,k} P\{|X_{nk}| \geqslant x\}$ for all $x \geqslant 0$. Then $R(0) = 1$ and $R(\cdot)$ is decreasing and left-continuous. Since $EX_{nk}^2 \leqslant K$, we have $R(x) \leqslant Kx^{-2}$ for $x > 0$. Hence, in particular, $R(x) \to 0$ as $x \to \infty$, and so there exists a nonnegative random variable W satisfying (2.7) with equality and $C = 1$. Moreover, $P\{W \geqslant x\} \leqslant Kx^{-2}$ and so, $EW^q \leqslant (2/(2-q))\, K^{q/2}$ for all $0 < q < 2$. In the same way, we see that there exist nonnegative random variables U_k and V_n satisfying (2.5) and (2.6) with equality and $C = 1$.

Corollary 1. Let $\{(X_{nk}, \mathscr{F}_{nk}), k \geqslant 1\}$ be sequences of square-integrable martingale differences satisfying $\sup_{n,k} E(X_{nk}^2 \mid \mathscr{F}_{n,k-1}) \leqslant K$ a.s. for some constant K. Let $p > 0$ and $\{a_{nk}\}$ be an array of real numbers such that

$$\sum_{k=1}^{\infty} a_{nk}^2 = O(n^{\delta}) \qquad \text{for some} \quad \delta < 2/p \tag{2.15}$$

If

$$A = \sup_{n,k \geqslant 1} |a_{nk}| \qquad \text{and} \qquad A_q = \sup_{n \geqslant 1} \sum_{k=1}^{\infty} |a_{nk}|^q, \qquad q > 0 \tag{2.16}$$

and U_k, V_n and W are chosen according to (2.5)–(2.7) respectively, then we have

$$\sum_{n=1}^{\infty} P\left\{\sup_{k \geqslant 1} \sum_{i=1}^{k} n^{-1/p} |a_{nk} X_{nk}| > \varepsilon\right\} < \infty \qquad \forall \varepsilon > 0 \tag{2.17}$$

in either of the following five cases:

$$\exists q > 0 \text{ so that } \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} n^{-q/p} |a_{nk}|^q E |X_{nk}|^q < \infty \tag{2.18}$$

$$p < 2 \text{ and } A_2 < \infty \tag{2.19}$$

$$A < \infty \text{ and } \sum_{k=1}^{\infty} E |U_k|^p < \infty \tag{2.20}$$

$$\exists q > 0 \text{ so that } A_q < \infty \text{ and } \sum_{n=1}^{\infty} n^{-q/p} E |V_n|^q < \infty \tag{2.21}$$

$$\exists q > 0 \text{ so that } A_q < \infty \text{ and } \begin{cases} E |W|^{\max(p, q)} < \infty, & \text{if } p \neq q \\ E |W|^p \log(1 + |W|) < \infty, & \text{if } p = q \end{cases} \tag{2.22}$$

The first two cases of this corollary generalize Theorem 3 of Li *et al.*[8] from independent to the martingale case. In fact, the case $p < 2$ is covered and the conclusion is also strengthened. On the other hand, Theorem 2 of Yu[11] is generalized in various directions. First, we allow a double array instead of a single martingale considered by Yu.[11] Secondly, no condition on the conditional pth moments are assumed as opposed to Yu;[11] the assumed conditions on the moments and conditional moments follow from the condition that $\sup_{n,k} E(|X_{nk}|^p \mid \mathscr{F}_{n,k-1}) \leqslant K$ a.s. (in the case $p \geqslant 2$). Thirdly, we need $\delta < 2/p$ only unlike $\delta < 1/p$ as required by Yu.[11] Finally, the assumption on the convergence of the infinite real series is weaker than Yu's[11] in the case of $p > 4$. In particular, a result of Chow[2] [see Theorem 1 of Yu[11]] for $p > 2$ follows as a very special case. The case $p = 2$ of Chow's result (and hence the original result of Hsu and Robbins[6]) is contained in Theorem 2.

Corollary 2. Let $\{(X_{nk}, \mathscr{F}_{nk}), k \geqslant 1\}$ be sequences of square-integrable martingale differences such that $\sup_{n,k \geqslant 1} E(X_{nk}^2 \mid \mathscr{F}_{n,k-1}) \leqslant K$ a.s. for some constant K. Let W be a random variable satisfying (2.7) and $\{a_{nk}\}$ be an array of real numbers. Let $p > 0$.

(a) Assume that $\sup_{n \geqslant 1} \sum_{k=1}^{\infty} |a_{nk}|^p < \infty$ and $E(|W|^p \log(1 + |W|)) < \infty$. Further, if $p > 2$, assume also that $\sum_{k=1}^{\infty} a_{nk}^2 = O(n^{\delta})$ for some $\delta < 2/p$, $q < p$. Then

$$\sum_{n=1}^{\infty} P\left\{n^{-1/p} \sup_{k \geqslant 1} \left| \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon \right\} < \infty \tag{2.23}$$

(b) If $E|W|^p < \infty$ and if for some $\delta < 2/p$, $q < p$, we have $\sum_{k=1}^{\infty} a_{nk}^2 = O(n^\delta)$ and $\sup_{n \geq 1} \sum_{k=1}^{\infty} |a_{nk}|^q < \infty$, then

$$\sum_{n=1}^{\infty} P\left\{n^{-1/p} \sup_{k \geq 1}\left|\sum_{i=1}^{k} a_{ni} X_i\right| > \varepsilon\right\} < \infty \tag{2.24}$$

Corollary 2 generalizes Theorem 2 of Li *et al.*[8] to the martingale case (consequently, also the aforementioned result of Chow in the case $p > 2$) and also removes a few restrictions. Part (I) of their theorem is extended from $p = 2$ to every $p > 0$ and from Part (II), the restrictions $p > 2$ and $q \geq 2$ are omitted.

Corollary 3. Let $\{(X_{nk}, \mathscr{F}_{nk}), k \geq 1\}$ be a sequence of martingale differences such that $\sup_{n,k} E(X_{n,k}^2 \mid \mathscr{F}_{n,k-1}) \leq K$ a.s. for some constant K. Let $c > 0$, $p, q \in \mathbb{R}$ be given numbers and let $\{b_{nk}\}$ be an array of real numbers satisfying

$$|b_{nk}| \leq c n^{-p} k^{-q} \qquad \forall n, \quad k \geq 1 \tag{2.25}$$

and

$$\sum_{k=1}^{\infty} b_{nk}^2 = O(n^{-\alpha}) \qquad \text{for some} \quad \alpha > 0 \tag{2.26}$$

If U_k, V_n and W are chosen according to (2.5)–(2.7) respectively, then we have

$$\sum_{n=1}^{\infty} P\left\{\sup_{k \geq 1}\left|\sum_{i=1}^{k} b_{ni} X_{ni}\right| > \varepsilon\right\} < \infty \qquad \forall \varepsilon > 0 \tag{2.27}$$

in any of the following seven cases:

$$\exists \mu > 0 \text{ so that} \sum_{n, k \geq 1, b_{nk} \neq 0} n^{-\mu p} k^{-\mu q} E|X_{nk}|^\mu < \infty \tag{2.28}$$

$$p > \tfrac{1}{2} \text{ and } q > \tfrac{1}{2} \tag{2.29}$$

$$p > 0 \text{ and } \sum_{k=1}^{\infty} k^{-q/p} E|U_k|^{1/p} < \infty \tag{2.30}$$

$$q > 0 \text{ and } \sum_{n=1}^{\infty} n^{-p/q} E|V_n|^{1/q} < \infty \tag{2.31}$$

$$p > 0,\ q > 0,\ p \neq q \text{ and } E\,|W|^{\max(p^{-1},\, q^{-1})} \tag{2.32}$$

$$p = q > 0 \text{ and } E(|W|^{1/p} \log(1 + |W|)) < \infty \tag{2.33}$$

$$p > |q|,\ |b_{nk}| = 0\ \forall k > n \text{ and } E\,|W|^{2/(p+q)} < \infty \tag{2.34}$$

Choosing $p = 1 + \beta$ and $q = -\beta$, the direct half of Theorem 4 of Li *et al.*[8] can now be readily extended to the martingale case, with a somewhat weaker set of conditions.

3. PROOFS

In this section, we present proofs of all the theorems of the previous section.

Proof of Theorem 1. (a) For $t > 0$, define a stopping time $T = \inf\{k \geqslant 1 : |S_k| \geqslant t\}$, where $\inf(\varnothing) = \infty$. Set $\|S\| = \sup_{k \geqslant 1} |S_k|$. Note that, by the assumption, $\{S_k, \mathscr{F}_k\}$ is an L^2-bounded martingale and so $\|S\| < \infty$ a.s. Given $\varepsilon > 0$, choose k_0 (depending on the sample point) such that $|S_{k_0}| > \|S\| - \varepsilon$. Thus on $\{T = l, \|S\| \geqslant t + r + s, \sup_{k \geqslant 1} |X_k| < s\}$, we have $\sup_{k \geqslant l} |S_k - S_l| \geqslant |S_{k_0} - S_l| \geqslant |S_{k_0}| - |S_{l-1}| - |X_l| > \|S\| - \varepsilon - t - s \geqslant r - \varepsilon$; the third inequality holds because $|S_{l-1}| < t$ on the relevant set. Since $\varepsilon > 0$ is arbitrary, $\sup_{k \geqslant l} |S_k - S_l| \geqslant r$. Hence

$$\begin{aligned} &P\{T = l, \|S\| \geqslant t + r + s\} \\ &\qquad \leqslant P\{T = l, \sup_{k \geqslant l} |S_k - S_l| \geqslant r\} + P\{T = l, \sup_{k \geqslant 1} |X_k| \geqslant s\} \end{aligned} \tag{3.1}$$

By (a conditional version of) Doob's inequality,

$$\begin{aligned} P\{T = l, \sup_{k \geqslant l} |S_k - S_l| \geqslant r\} &= \int_{\{T = l\}} P\{\sup_{k \geqslant l} |S_k - S_l| \geqslant r \mid \mathscr{F}_l\}\, dP \\ &\leqslant r^{-2} \int_{\{T = l\}} \left[\sum_{k \geqslant l+1} E(X_k^2 \mid \mathscr{F}_l) \right] dP \\ &\leqslant r^{-2} \int_{\{T = l\}} \left[\sum_{k=1}^{\infty} E(X_k^2 \mid \mathscr{F}_{k-1}) \right] dP \\ &\leqslant M r^{-2} P\{T = l\} \end{aligned} \tag{3.2}$$

Combining with (3.1) and summing over l, (2.1) follows since $\sum_{l=1}^{\infty} P\{T = l\} \leqslant P\{\|S\| \geqslant t\}$.

(b) For $m=1$, the result is true since by Doob's inequality,

$$P\{\sup_{k\geqslant 1}|S_k|\geqslant 2t\}\leqslant\frac{1}{4t^2}\sum_{k=1}^{\infty}EX_k^2\leqslant\frac{M}{4t^2} \tag{3.3}$$

For an $m\geqslant 1$, by Part (a),

$$\begin{aligned}P\{\sup_{k\geqslant 1}|S_k|\geqslant 2(m+1)\,t\} &= P\{\sup_{k\geqslant 1}|S_k|\geqslant 2mt+t+t\}\\ &\leqslant P\{\sup_{k\geqslant 1}|X_k|\geqslant t\}+\frac{M}{t^2}P\{\sup_{k\geqslant 1}|S_k|\geqslant 2mt\}\end{aligned} \tag{3.4}$$

The proof can now be completed by induction. □

Proof of Theorem 2. Find a positive integer $m\geqslant\lambda$ and apply Part (b) of Theorem 1 with $t=\varepsilon/(2m)$. □

Proof of Theorem 3. By an application of Theorem 2 to the array $b_{nk}X_{nk}$ with $c_n=1$, it is clear that (2.13) implies (2.14). The rest follows from the following easy estimates: For any nonnegative random variable Z,

$$E\xi_{\bullet k}(Z/\varepsilon)\geqslant\sum_{n=1}^{\infty}P\{|b_{nk}Z|>\varepsilon\}$$

$$E\xi_{n\bullet}(Z/\varepsilon)\geqslant\sum_{k=1}^{\infty}P\{|b_{nk}Z|>\varepsilon\}$$

$$E\xi_{\bullet\bullet}(Z/\varepsilon)\geqslant\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}P\{|b_{nk}Z|>\varepsilon\}$$

$$E\xi(Z/\varepsilon)\geqslant P\{Z>\varepsilon\}$$ □

Proof of Corollary 1. We set $b_{nk}=n^{-1/p}a_{nk}$ and note that $\sum_{k=1}^{\infty}b_{nk}^2=O(n^{-\alpha})$, where $\alpha=(2/p)-\delta$. We now apply Theorem 3 with $\xi(x)=x^q$ for $x>1$ and $\xi(x)=0$ otherwise. We need the following estimates (i)–(iii) which are easy to verify. First note that $A<\infty$ if for some $q>0$, $A_q<\infty$. We shall use this again in the proof of Corollary 2.

(i) If $A<\infty$, then

$$\xi_{\bullet k}(x)\leqslant A^q x^q\sum_{n<|Ax|^p}n^{-q/p}\leqslant\begin{cases}Cx^{\max(p,q)}, & \text{if } p\neq q\\ Cx^p\log(1+x), & \text{if } p=q\end{cases}$$

where C is a constant depending only on (p, q, A).

(ii) If $A_q < \infty$, $\xi_{n\bullet}(x) \leqslant A_q n^{-q/p} x^q$.

(iii) If $A_q < \infty$, then

$$\xi_{\bullet\bullet}(x) \leqslant A_q x^q \sum_{n < |Ax|^q} n^{-q/p} \leqslant \begin{cases} Cx^{\max(p,q)}, & \text{if } p \neq q \\ Cx^p \log(1+x), & \text{if } p = q \end{cases}$$

where C is a constant depending only on (p, q, A, A_q).

If (2.18) holds, then (2.9) verifies. If $p < 2$, then (2.18) holds for $q = 2$, since EX_{nk}^2 is bounded and $n^{-2/p}$ is summable. If (2.20) holds, apply (i) with any $q < p$ to verify (2.10). When (2.21) holds, (2.11) can be verified by an application of (ii). Finally, if (2.22) holds, it is immediate from the estimate (iii) that (2.12) also holds. This completes the proof. □

Proof of Corollary 2. If $p \leqslant 2$ and $\sup_{n \geqslant 1} \sum_{k=1}^{\infty} |a_{nk}|^p < \infty$, then (2.15) trivially holds. Now we can verify (2.22) of Corollary 1. For case (a), we choose $q = p$ whereas for case (b), we apply (2.22) with the same q. □

Proof of Corollary 3. If (2.28) holds, then (2.9) holds with $\xi(x) = x^{\mu}$. The next case is easily resolved as a special case by taking $\mu = 2$. For the rest of the cases, we choose $\xi(x)$ to be the indicator of the interval $(1, \infty)$. Then the following estimates can be established:

$$p > 0 \Rightarrow \xi_{\bullet k}(x) \leqslant c^{1/p} k^{-q/p} x^{1/p}$$

$$q > 0 \Rightarrow \xi_{n\bullet}(x) \leqslant c^{1/q} n^{-p/q} x^{1/q}$$

$$p > 0, \quad q > 0 \Rightarrow \xi_{\bullet\bullet}(x) \leqslant \begin{cases} Cx^{\max(p^{-1}, q^{-1})}, & \text{if } p \neq q \\ Cx^{1/p} \log(1+x), & \text{if } p = q \end{cases}$$

Moreover, if $b_{nk} = 0$ for all $k > n$ and $|q| < p$, then

$$\xi_{\bullet\bullet}(x) \leqslant Cx^{2/(p+q)}$$

It is now easy to see that the result follows from various cases of Theorem 3. □

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