Acta Math. Hungar.
71 (4) (1996), 327–336.

EXTENSIONS OF THE STRONG LAW OF LARGE NUMBERS OF MARCINKIEWICZ AND ZYGMUND FOR DEPENDENT VARIABLES

T. K. CHANDRA and S. GHOSAL (Calcutta)

The classical strong law of large numbers (SLLN) due to Kolmogorov has been extended recently to various weakly dependent random variables (rvs) which are not necessarily identically distributed. It is natural to enquire whether the SLLN of Marcinkiewicz and Zygmund (MZSLLN) (see Theorem 3.2.3 of Stout [11]) holds under similar relaxed conditions. In this paper, we try to fill this gap and show that this weakening is indeed possible; the independence assumption is relaxed to *φ-mixing* or *asymptotically almost negatively associated* sequences defined below. Related results are established for *asymptotically quadrant sub-independent* (AQSI) sequences (see Definition 1). These concepts are interesting as they unify, to some extent, the notion of mixing-type sequences and that of negatively dependent sequences.

In contrast to the usual proofs of the MZSLLN (see, e.g., Stout [11] and Chow and Teicher [5]), our proofs use *maximal inequalities* at a crucial step. We are thus able to relax *mutual independence* to a great extent, and the assumption of *identical distribution* is also relaxed considerably. This technique is likely to be useful for other dependences also, provided suitable maximal inequalities are available under those dependences.

DEFINITION 1. A sequence $\{X_n\}$ of rvs is called *asymptotically quadrant sub-independent* (AQSI) if there exists a nonnegative sequence $\{q(m)\}$ such that for all $i \neq j$,

$$P\{X_i > s, X_j > t\} - P\{X_i > s\}P\{X_j > t\} \leqq q\big(|i-j|\big)\,\alpha_{ij}(s,t), s,t > 0,$$

$$P\{X_i < s, X_j < t\} - P\{X_i < s\}P\{X_j < t\} \leqq q\big(|i-j|\big)\,\beta_{ij}(s,t), s,t < 0,$$

where $q(m) \to 0$ and $\alpha_{ij}(s,t) \geqq 0$, $\beta_{ij}(s,t) \geqq 0$.

Note that pairwise negative quadrant dependent, pairwise m-dependent and AQI rvs (introduced by Birkel [2]; see Definition 2 below) are special cases of AQSI rvs.

DEFINITION 2. A sequence $\{X_n\}$ of rvs is called *asymptotically quadrant independent* (AQI) if there exists a nonnegative sequence $\{q(m)\}$ such that for all $i \neq j$ and $s,t \in \mathbf{R}$,

$$\big|P\{X_i > s, X_j > t\} - P\{X_i > s\}P\{X_j > t\}\big| \leqq q\big(|i-j|\big)\,\alpha_{ij}(s,t),$$

$$\left| P\{X_i < s, X_j < t\} - P\{X_i < s\}P\{X_j < t\} \right| \leqq q\big(|i-j|\big)\beta_{ij}(s,t),$$

where $q(m) \to 0$ and $\alpha_{ij}(s,t) \geqq 0$, $\beta_{ij}(s,t) \geqq 0$.

For a sequence $\{X_n\}$ of rvs, put $S_n = \sum_{i=1}^n X_i$, $n \geqq 1$; also, define $G(y) = \sup_{n\geqq 1} n^{-1} \sum_{k=1}^n P\{|X_k| > y\}$. Finally, C will stand for a generic constant.

Our proof of MZSLLN needs that the sequence $\{X_n\}$ should satisfy a dependence condition which is well behaved in the sense that it admits a maximal inequality for the truncated rvs. Matula [8] has recently established a maximal inequality for negatively associated (NA) sequences; consequently this allows us to extend MZSLLN from independent to NA sequences. Recall that a sequence $\{X_n\}$ is called NA if for every finite disjoint subsets $A, B \subset \{1,2,\ldots\}$, and coordinatewise increasing functions $f : \mathbf{R}^A - \mathbf{R}$ and $g : \mathbf{R}^B \to \mathbf{R}$, $\operatorname{cov}\big(f(X_i : i \in A), g(X_i : i \in B)\big) \leqq 0$, whenever it exists. By inspecting the proof of Matula's [2] maximal inequality, we shall see that *one can also allow positive correlations provided they are small*. Primarily motivated by this, we introduce the following dependence condition:

DEFINITION 3. A sequence $\{X_n\}$ of rvs is called *asymptotically almost negatively associated* (AANA) if there is a nonnegative sequence $q(m) \to 0$ such that

$$\operatorname{cov}\big(f(X_m), g(X_{m+1},\ldots,X_{m+k})\big) \leqq q(m)\big(\operatorname{var}\big(f(X_m)\big)\operatorname{var}\big(g(X_{m+1},\ldots,X_{m+k})\big)\big)^{1/2} \tag{1}$$

for all $m, k \geqq 1$ and for all coordinatewise increasing continuous functions f and g whenever the right side of (1) is finite.

The family of AANA sequences contains NA (in particular, independent) sequences and some more sequences of rvs which are not much deviated from being negatively associated. Condition (1) is clearly satisfied if the $R_{2,2}$-measure of dependence (see Bradley et al. [1]) between $\sigma\langle X_m\rangle$ and $\sigma\langle X_{m+1}, X_{m+2}, \ldots\rangle$ converges to zero. The following is a non-trivial example of an AANA sequence. It is possible to construct similar examples, but we shall not discuss this topic any more here.

EXAMPLE. Let $\{Y_n\}$ be i.i.d. $N(0,1)$ variables and define $X_n = (1 + a_n^2)^{-1/2}(Y_n + a_n Y_{n+1})$ where $a_n > 0$ and $a_n \to 0$. Note that $\{X_n\}$ is not NA (indeed, is associated and 1-dependent). We shall show that the correlation coefficient between $U := f(X_m)$ and $V := g(X_{m+1},\ldots,X_{m+k})$ is dominated in absolute value by a_m. It suffices to prove this under the additional hypotheses $EU = 0 = EV$, $EU^2 = 1 = EV^2$. Then

$$\big(\operatorname{cov}(U,V)\big)^2 \leqq \big(\operatorname{cov}\big(U, E(U|X_{m+1},\ldots,X_{m+k})\big)\big)^2$$

$$= E\big(E(U|X_{m+1},\dots,X_{m+k})\big)^2 \leqq E\big(E(U|Y_{m+1},\dots,Y_{m+k+1})\big)^2$$

$$= E\big(E(U|Y_{m+1})\big)^2 = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)\left(\frac{\psi_m(x,y)}{\phi(x)}-1\right)\phi(x)\,dx\right)^2\phi(y)\,dy$$

where $\psi_m(x,y)$ is the conditional density of X_m given $Y_{m+1}=y$ and $\phi(x)$ is the density of $N(0,1)$. By Cauchy–Schwarz inequality, the last integral is at most

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\psi_m(x,y)}{\phi(x)}-1\right)^2\phi(x)\,dx\,\phi(y)\,dy = a_m^2.$$

THEOREM 1. *Let $X_1,\dots,X_n$ be mean zero, square integrable rvs such that* (1) *holds for $1\leqq m<k+m\leqq n$ and for all coordinatewise increasing continuous functions f and g whenever the right side of* (1) *is finite. Let $A^2=\sum_{m=1}^{n-1}q^2(m)$ and $\sigma_k^2=EX_k^2$, $k\geqq 1$. Then*

$$P\Big\{\max_{1\leqq k\leqq n}|S_k|\geqq\varepsilon\Big\}\leqq 2\varepsilon^{-2}\left(A+(1+A^2)^{1/2}\right)^2\sum_{k=1}^{n}\sigma_k^2. \tag{2}$$

PROOF. The proof is based on certain ideas of Matula [8]. Clearly,

$$P\Big\{\max_{1\leqq k\leqq n}|S_k|\geqq\varepsilon\Big\}\leqq P\Big\{\max_{1\leqq k\leqq n}S_k\geqq\varepsilon\Big\}+P\Big\{\max_{1\leqq k\leqq n}(-S_k)\geqq\varepsilon\Big\}.$$

Set $T_k=\max(X_k,X_k+X_{k+1},\dots,X_k+\dots+X_n)$, $1\leqq k\leqq n$. We have $T_k = X_k+T_{k+1}^+$ and consequently $ET_k^2\leqq\sigma_k^2+ET_{k+1}^2+2q(k)\sigma_k(ET_{k+1}^2)^{1/2}$, $1\leqq k\leqq n-1$. Define $\{\xi_k\}$ by $\xi_n^2=\sigma_n^2$ and $\xi_k^2=\sigma_k^2+\xi_{k+1}^2+2q(k)\sigma_k\xi_{k+1}$, $1\leqq k\leqq n-1$. Clearly, $ET_k^2\leqq\xi_k^2$ and $\{\xi_k\}$ is decreasing, whence $\xi_k^2\leqq\sigma_k^2+\xi_{k+1}^2+2q(k)\sigma_k\xi_1$, $1\leqq k\leqq n-1$. Substituting sequentially and using the Cauchy–Schwarz inequality, we get

$$\xi_1^2\leqq\sum_{k=1}^{n}\sigma_k^2+2\xi_1\sum_{k=1}^{n-1}q(k)\sigma_k\leqq\tau^2+2\xi_1A\tau$$

where $\tau^2=\sum_{k=1}^{n}\sigma_k^2$. Hence $(\xi_1-A\tau)^2\leqq\tau^2(1+A^2)$, which implies that $ET_1^2\leqq\xi_1^2\leqq\tau^2\big(A+(1+A^2)^{1/2}\big)^2$ and so (2) follows. □

REMARK 1. Theorem 1 extends Lemma 4 of Matula [8]. The proof of Theorem 3 of Matula [8] can be used to show that his Theorem 3 and its corollary remain true if the assumption of "negatively associated rvs" is relaxed to "AANA rvs with $\sum_{k=1}^{\infty}q^2(k)<\infty$".

The following result will be used in the sequel.

LEMMA 1. *Let $\{X_n\}$ be any sequence of rvs satisfying*

$$(3)\quad \int_0^\infty y^{p-1}G(y)\,dy<\infty,\quad \sum_{k=1}^\infty P\{|X_k|^p>k\}<\infty \quad \textit{for some } 0<p<2.$$

Then

(a) $\sum_{k=1}^\infty k^{-2/p}E\left(X_k^2I\{|X_k|^p\leqq k\}\right)<\infty$.

(b) $\sum_{k=1}^\infty k^{-1/p}E\left(|X_k|I\{|X_k|^p\leqq k\}\right)<\infty$ *if* $0<p<1$,
$n^{-1/p}\sum_{k=1}^n E\left(|X_k|I\{|X_k|^p>k\}\right)\to 0$ *if* $1\leqq p<2$.

(c) $\sum_{n=1}^\infty 2^{-2n/p}\sum_{k=1}^{2^n}E\left(X_k^2I\{|X_k|^p\leqq k\}\right)<\infty$.

(d) $\sum_{n=1}^\infty 2^{-2n/p}\sum_{k=1}^{2^n}k^{2/p}P\{|X_k|^p>k\}<\infty$.

PROOF. (a) The given expression is dominated by

$$C\sum_{k=1}^\infty\sum_{j=k}^\infty j^{-2/p-1}E\left(X_k^2I\{|X_k|^p\leqq k\}\right)$$

$$\leqq C\sum_{k=1}^\infty\sum_{j=k}^\infty j^{-2/p-1}\int_0^{k^{1/p}}yP\{|X_k|>y\}\,dy$$

$$=C\sum_{j=1}^\infty\sum_{k=1}^j j^{-2/p-1}\sum_{n=1}^k\int_{(n-1)^{1/p}}^{n^{1/p}}yP\{|X_k|>y\}\,dy$$

$$\leqq C\sum_{j=1}^\infty\sum_{n=1}^j j^{-2/p}\int_{(n-1)^{1/p}}^{n^{1/p}}y\left(j^{-1}\sum_{k=1}^j P\{|X_k|>y\}\right)dy$$

$$\leqq C\sum_{n=1}^\infty\sum_{j=n}^\infty j^{-2/p}\int_{(n-1)^{1/p}}^{n^{1/p}}yG(y)\,dy\leqq C\sum_{n=1}^\infty n^{1-2/p}\int_{(n-1)^{1/p}}^{n^{1/p}}yG(y)\,dy$$

$$\leqq C\sum_{n=1}^\infty\int_{(n-1)^{1/p}}^{n^{1/p}}y^{p-1}G(y)\,dy<\infty.$$

(b) The proof of (b), in the case $0<p<1$, is similar to that of (a). Now let $1\leqq p<2$ and fix $N\geqq 1$. For $n>N$, we have

$$n^{-1/p}\sum_{k=1}^n E\left(|X_k|I\{|X_k|^p>k\}\right)$$

$$= n^{-1/p}\sum_{k=1}^{N}\int_{k^{1/p}}^{\infty}P\{|X_k|>y\}\,dy + n^{-1/p}\sum_{k=N+1}^{n}\int_{k^{1/p}}^{\infty}P\{|X_k|>y\}\,dy$$

$$+n^{-1/p}\sum_{k=1}^{n}k^{1/p}P\{|X_k|^p>k\}.$$

Obviously, the first term on the right side goes to 0 as $n\to\infty$; the third term converges to 0 by (3) and Kronecker's lemma. Finally, the second term is at most

$$n^{-1/p}\sum_{k=N}^{n}\sum_{j=k}^{\infty}\int_{j^{1/p}}^{(j+1)^{1/p}}P\{|X_k|>y\}\,dy$$

$$\leqq n^{-1/p}\sum_{j=N}^{\infty}\int_{j^{1/p}}^{(j+1)^{1/p}}\sum_{k=1}^{\min(j,n)}P\{|X_k|>y\}\,dy$$

$$\leqq n^{-1/p}\sum_{j=N}^{\infty}\Big(\big(\min(j,n)\big)^{1/p}\Big)^{(p-1)+1}\int_{j^{1/p}}^{(j+1)^{1/p}}G(y)\,dy$$

$$\leqq \sum_{j=N}^{\infty}\int_{j^{1/p}}^{(j+1)^{1/p}}y^{p-1}G(y)\,dy\to 0 \quad\text{as}\quad N\to\infty.$$

Part (c) follows from (a) and an interchange of summation signs. Part (d) follows similarly. □

REMARK 2. The condition in (3) is obviously satisfied if $\{|X_n|\}$ is stochastically dominated by a rv X with $EX^p<\infty$.

REMARK 3. If $0<p<1$ and (3) holds, then $\sum_{k=1}^{\infty}k^{-1/p}|X_k|<\infty$ a.s.; hence the MZSLLN holds *irrespective of any dependence condition.* To see this, note that it is enough to verify that

$$Y:=\sum_{k=1}^{\infty}k^{-1/p}|X_k|I\{|X_k|^p\leqq k\}<\infty \quad\text{a.s.};$$

this is true, since $EY<\infty$ by Lemma 1(b) and the monotone convergence theorem.

Henceforth we shall assume $1\leqq p<2$ *and* $EX_n=0$ *for all* n.

LEMMA 2. *Let $\{X_n\}$ be a sequence of rvs satisfying* (3) *and define*

$$Y_n = X_n I\{|X_n| \leqq n^{1/p}\} + n^{1/p} I\{X_n > n^{1/p}\} - n^{1/p} I\{X_n < -n^{1/p}\}, \quad n \geqq 1.$$

Assume that there exists K such that

$$\operatorname{var}\Big(\sum_{k=1}^{n} Y_k\Big) \leqq K \sum_{k=1}^{n} (1 + EY_k^2) \quad \textit{for all } n \geqq 1.$$

Then

$$2^{-n/p} \sum_{k=1}^{2^n} (Y_k - EY_k) \to 0 \quad a.s.$$

PROOF. This follows because $\sum_{n=1}^{\infty} 2^{-2n/p} \operatorname{var}\Big(\sum_{k=1}^{2^n} Y_k\Big)$ is at most

$$K \sum_{n=1}^{\infty} 2^{-2n/p} \sum_{k=1}^{2^n} \Big(1 + E\left(X_k^2 I\{\,|X_k|^p \leqq k\}\right) + k^{2/p} P\{\,|X_k|^p > k\}\Big) < \infty,$$

by Lemma 1(c) and (d). □

REMARK 4. Lemma 2 also holds if Y_n is redefined as $X_n I\{\,|X_n|^p \leqq n\}$.

THEOREM 2. *Let $\{X_n\}$ be an AANA sequence satisfying* (3) *and let* $B^2 := \sum_{m=1}^{\infty} q^2(m) < \infty$. *Then $n^{-1/p} S_n \to 0$ a.s.*

PROOF. Define Y_n as in Lemma 2 and put $U_n = Y_n - EY_n$, $T_n = T(n) = \sum_{k=1}^{n} U_k$. It suffices to show that $n^{-1/p} T_n \to 0$ a.s. Clearly, $\{U_n\}$ is also AANA with the same $q(\cdot)$. So

$$ET_k^2 \leqq EU_k^2 + ET_{k-1}^2 + 2q(k)\left(EU_k^2 ET_{k-1}^2\right)^{1/2}, \quad 2 \leqq k \leqq n.$$

Proceeding as in the last part of the proof of Theorem 1, we get

$$ET_n^2 \leqq \left(B + (1 + B^2)^{1/2}\right)^2 \sum_{k=1}^{n} EU_k^2.$$

Therefore, Lemma 2 yields that $2^{-n/p} T(2^n) \to 0$ a.s. For each m, let $n = n_m$ be such that $2^n \leqq m < 2^{n+1}$. It remains to show that $Z_m := m^{-1/p} \sum_{i=2^n+1}^{m} U_i \to 0$ a.s. The proof is therefore complete, since by Theorem 1 and Lemma 1(c),(d),

$$\sum_{n=1}^{\infty} P\{\max(|Z_m| : 2^n \leqq m < 2^{n+1}) > \varepsilon\} < \infty. \tag{4}$$ □

We now recall some definitions involving mixing concepts.

DEFINITION 4. For two sub σ-fields $\mathcal{F}$ and $\mathcal{G}$ of a probability space $(\Omega, \mathcal{A}, P)$, define

$$\varphi(\mathcal{F}, \mathcal{G}) = \sup\left\{ \left| P(G|F) - P(G) \right| : F \in \mathcal{F}, G \in \mathcal{G}, P(F) > 0 \right\},$$

$$\alpha(\mathcal{F}, \mathcal{G}) = \sup\left\{ \left| P(F \cap G) - P(F)P(G) \right| : F \in \mathcal{F}, G \in \mathcal{G} \right\},$$

$$\psi(\mathcal{F}, \mathcal{G}) = \sup\left\{ \left| \frac{P(F \cap G)}{P(F)P(G)} - 1 \right| : F \in \mathcal{F}, G \in \mathcal{G}, P(F)P(G) > 0 \right\}.$$

For a stochastic sequence $\{X_n\}_{1 \leqq n < \infty}$, let $\mathcal{F}_n^m$ be the σ-field generated by $\{X_n, \ldots, X_m\}, n \leqq m$. We define

$$\varphi_m = \sup_n \varphi(\mathcal{F}_1^n, \mathcal{F}_{n+m}^{n+m}), \ \alpha_m = \sup_n \alpha(\mathcal{F}_1^n, \mathcal{F}_{n+m}^{n+m}), \ f_m = \sup_n \psi(\mathcal{F}_1^n, \mathcal{F}_{n+m}^{n+m}).$$

The sequence $\{X_n\}$ is called φ-mixing (respectively, α-mixing or $*$-mixing) if $\varphi_m \to 0$ (respectively, $\alpha_m \to 0$ or $f_m \to 0$) as $m \to \infty$.

THEOREM 3. *Assume that $\{X_n\}$ is φ-mixing with $\sum_{n=1}^{\infty} \varphi_n^{1/2} < \infty$ and that (3) holds. Then $n^{-1/p} S_n \to 0$ a.s.*

PROOF. We use the terminologies of McLeish [9]. Define $Y_n = X_n I\{ |X_n|^p \leqq n\}$. As $\{\varphi_n\}$ is trivially of size -1, Theorem 2.7 of McLeish [9] implies that *the $U_n := Y_n - EY_n$ form an L^2-mixingale (difference) sequence with $\psi_m = \varphi_m^{1/2}$, $c_i = \|U_i\|_2$ and $\{\psi_m\}$ is of size $-1/2$. (We need the mixing condition to deduce only this fact.)* Also, for $i < j$

$$EU_iU_j = E\big(U_i E(U_j | U_1, \ldots, U_i)\big) \leqq \psi_{j-i} \|U_i\|_2 \|U_j\|_2.$$

By straightforward arguments (see, e.g., Lemma 1 of Chandra [3]) and Remark 4, it follows that $2^{-n/p} \sum_{k=1}^{2^n} U_k \to 0$ a.s. Rest of the proof is completed as in Theorem 2 by appealing to the maximal inequality of McLeish [9]. □

An important example where Theorem 3 can be applied is a stationary Markov sequence satisfying Doeblin's condition; see, e.g., Doob ([6], Ch. 5) and Rosenblatt ([10], Ch. 7).

THEOREM 4. *Assume that $\{X_n\}$ is AQSI with $\sum_{m=1}^{\infty} q(m) < \infty$, and for all $i \neq j$*

$$\left\{ \begin{aligned} &\int_0^{j^{1/p}} \int_0^{i^{1/p}} \alpha_{ij}(s,t)\,ds\,dt \leqq D(1 + EY_i^2 + EY_j^2), \\ &\int_0^{j^{1/p}} \int_0^{i^{1/p}} \beta_{ij}(s,t)\,ds\,dt \leqq D(1 + EY_i^2 + EY_j^2) \end{aligned} \right. \tag{5}$$

where $\{Y_n\}$ *is as in Lemma* 2 *and* D *is a constant, and that* (3) *holds. Then* $n^{-1/p}(\log n)^{-1}S_n \to 0$ *a.s.*

PROOF. Clearly, $\{Y_n\}$ forms an AQSI sequence. Now by Lemma 2 of Lehmann [7] $\operatorname{cov}(Y_i^+, Y_j^+) \leqq Dq(|i-j|)(1+EY_i^2+EY_j^2)$; so, as in the proof of Theorem 3, $\operatorname{var}\left(\sum_{i=1}^n Y_i^+\right) \leqq C\sum_{i=1}^n(1+EY_i^2)$ for all n. Similarly, $\operatorname{var}\left(\sum_{i=1}^n Y_i^-\right) \leqq C\sum_{i=1}^n(1+EY_i^2)$ for all n. Thus

$$\operatorname{var}\left(\sum_{i=1}^n Y_i\right) \leqq 2\operatorname{var}\left(\sum_{i=1}^n Y_i^+\right) + 2\operatorname{var}\left(\sum_{i=1}^n Y_i^-\right) \leqq C\sum_{i=1}^n(1+EY_i^2)$$

for all n. Hence Lemma 2 implies that $2^{-n/p}\sum_{k=1}^{2^n} U_k \to 0$ a.s. where $U_n = Y_n - EY_n$. Consider the variable $Z_m^* = m^{-1/p}(\log m)^{-1}\sum_{i=2^n+1}^m U_i$ where $n = n_m$ is as in the proof of Theorem 2. Thus it remains to verify (4) (with Z_m replaced by Z_m^*). Now the left side of (4) is dominated by

$$2D\sum_{k=1}^\infty q(k)\sum_{n=1}^\infty (n+3)^2 2^{-2n/p}(n\log 2)^{-2}\sum_{k=2^n+1}^{2^{n+1}}(1+EY_i^2)$$

$$\leqq C\sum_{n=1}^\infty 2^{-2n/p}\sum_{i=1}^{2^{n+1}}(1+EY_i^2) < \infty$$

(by Lemma 3 below and Lemma 1(c) and (d)). □

LEMMA 3. *Let* $X_1, \dots, X_n$ *be square integrable rvs and let there exist* $a_1^2, \dots, a_n^2$ *satisfying* $E(X_{m+1}+\dots+X_{m+p})^2 \leqq a_{m+1}^2+\dots+a_{m+p}^2$ *for all* $m, p \geqq 1$, $m+p \leqq n$. *Then we have*

$$E\left(\max_{1\leqq k\leqq n}\left(\sum_{i=1}^k X_i\right)^2\right) \leqq \left((\log n/\log 3)+2\right)^2\sum_{i=1}^n a_i^2.$$

This is an extension of the well-known Rademacher–Mensov inequality; a proof can be found in Chandra and Ghosal ([4], Theorem 10).

One can now guess that the following variation of Theorem 4 is true.

THEOREM 5. *Assume that* $\{X_n\}$ *is AQSI with* $\sum_{m=1}^{\infty} q(m) < \infty$, (5) *holds and that*

$$(6) \quad \int_0^{\infty} y^{p-1}(\log y)^2 G(y)\,dy < \infty, \qquad \sum_{n=1}^{\infty} (\log n)^2 P\{\,|X_n|^p > n\} < \infty.$$

Then $n^{-1/p}S_n \to 0$ *a.s.*

PROOF. Define Y_n as in Theorem 4 and proceed as in its proof. It, therefore, remains to show that $\sum_{n=1}^{\infty} n^2 2^{-2n/p} \sum_{k=2^n+1}^{2^{n+1}} EU_k^2 < \infty$. In fact, as in Lemma 1, we have

$$\sum_{k=1}^{\infty} k^{-2/p}(\log k)^2 E\left(X_k^2 I\{\,|X_k|^p \leqq k\}\right)$$

$$\leqq C \sum_{j=1}^{\infty} j^{-2/p}(\log j)^2 \sum_{n=1}^{j} \int_{(n-1)^{1/p}}^{n^{1/p}} yG(y)\,dy$$

$$\leqq C \sum_{n=1}^{\infty} n^{1-2/p}(\log n)^2 \int_{(n-1)^{1/p}}^{n^{1/p}} yG(y)\,dy < \infty$$

since $y^{1-2/p}(\log y)^2$ is eventually decreasing. □

REMARK 5. If in Theorem 5 we have pairwise independence in place of AQSI, then the second part of (6) can be relaxed to the second part of (3) since we can then redefine Y_n as $X_n I\{\,|X_n|^p \leqq n\}$.

REMARK 6. The following is a sufficient condition for (5): For all $i \neq j$,

$$P\{X_i > s, X_j > t\} \leqq \left(1 + q(|i-j|)\right) P\{X_i > s\}P\{X_j > t\}, \quad s,t > 0$$

$$P\{X_i < s, X_j < t\} \leqq \left(1 + q(|i-j|)\right) P\{X_i < s\}P\{X_j < t\}, \quad s,t < 0,$$

and $\sum_{m=1}^{\infty} q(m) < \infty$. This condition is much weaker than demanding that $\{X_n\}$ is $*$-mixing with coefficients $\{q(m)\}$. Another sufficient condition for (5) can be obtained by modifying Example 1 of Birkel [2]: Let $\{X_n\}$ be α-mixing with the coefficient $\alpha(m)$ and let $1 < p, r < \infty$ with $2/p + 1/r = 1$ be such that $\sum_{m=1}^{\infty} \alpha(m)^{1/p} < \infty$, $\sup_{i \geqq 1} E|X_i|^k < \infty$ for some $k > p$. Then (5) holds with Birkel's [2] choices of $q(m)$, $\alpha_{ij}(s,t)$, $\beta_{ij}(s,t)$ and $q = p$.

REMARK 7. It will be of some interest to investigate whether the "log terms" are really necessary in Theorems 4 and 5, at least for pairwise independent rvs.

References

[1] R. C. Bradley, W. Bryc and S. Janson, On dominations between measures of dependence, *J. Mult. Anal.*, **3** (1987), 312–329.
[2] T. Birkel, Laws of large numbers under dependence assumptions, *Statist. Probab. Lett.*, **7** (1992), 17–20.
[3] T. K. Chandra, Extensions of Rajchman's strong law of large numbers, *Sankhyā, Ser.* A **53** (1991), 118–121.
[4] T. K. Chandra and S. Ghosal, Some elementary strong laws of large numbers: a review, *Tech. Report, Indian Statistical Institute* (1993).
[5] Y. S. Chow and H. Teicher, *Probability Theory: Indepedence, Interchangeability, Martingales*, Second Edition, Springer-Verlag (New York, 1988).
[6] J. L. Doob, *Stochastic Processes*, John Wiley (New York, 1953).
[7] E. L. Lehmann, Some concepts of dependence, *Ann. Math. Statist.*, **43** (1966), 1137–1153.
[8] P. Matula, A note on the almost sure convergence of sums of negatively dependent random variables, *Statist. Probab. Lett.*, **15** (1992), 209–213.
[9] D. L. McLeish, A maximal inequality and dependent strong laws, *Ann. Probab.*, **3** (1975), 829–839.
[10] M. Rosenblatt, *Markov Processes, Structure and Asymptotic Behavior*, Springer-Verlag (New York, 1971).
[11] W. F. Stout, *Almost Sure Convergence*, Academic Press (New York, 1974).

(Received October 18, 1994; revised February 20, 1995)

DIVISION OF THEORETICAL
STATISTICS AND MATHEMATICS
INDIAN STATISTICAL INSTITUTE
203, BARRACKPUR TRUNK ROAD
CALCUTTA - 700 035
INDIA