**Project 1 of ST 755, Fall 2011**

**Due: Monday, 9/26/2011**

1. In class, we discussed the following mixed model equation for $Y = X\beta + Zb + e$ where $b \sim \mathrm{N}(0, G)$ and $e \sim \mathrm{N}(0, R)$:

$$\begin{bmatrix} X^T R^{-1} X & X^T R^{-1} Z \\ Z^T R^{-1} X & Z^T R^{-1} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \beta \\ b \end{bmatrix} = \begin{bmatrix} X^T R^{-1} Y \\ Z^T R^{-1} Y \end{bmatrix}.$$

Using matrix algebra, show without the normality distributional assumption for $b$ and $e$ that the above equation gives the MLE of $\beta$ and BLUP for $b$:

$$\widehat{\beta} = (X^T V^{-1} X)^{-1} X^T V^{-1} Y, \;\; \widehat{b} = G Z^T P Y.$$

2. (The paired t-test problem) Consider the following model

$$Y_{ij} = \alpha_i + x_j \beta_x + e_{ij}, \quad i = 1, ..., m, \quad j = 1, 2,$$

where $\beta_x$ is the parameter of major interest, $x_j = I(j = 1)$ and $e_{ij} \sim N(0, \sigma^2)$. Treat $\alpha_i$ as fixed parameters. Find the MLEs of $\beta_x, \sigma^2$, and the REML estimator of $\sigma^2$. Comment on the large sample properties of the MLE and the REML estimator of $\sigma^2$. Also find the variance of the MLE of $\beta_x$. Comment on the inference on $\beta_x$ using the MLE and the REML estimator of $\sigma^2$.

3. Consider the following Laird-Ware model for repeated data:

$$Y_i = X_i \beta + Z_i b_i + e_i, \quad i = 1, ..., m,$$

where $Y_i$ is an $n_i \times 1$ vector of data from subject $i$, $\beta$ is the fixed effects, $b_i \overset{i.i.d.}{\sim} N(0, D_{k\times k})$ is the subject-specific random effects with unstructured variance matrix $D_{k\times k}$, and $e_i \sim N(0, \sigma^2 I_{n_i \times n_i})$. It is further assumed that $b_i$ and $e_i$ are independent.

Let us consider a situation where $k$ is small to moderate but $n_i$'s are so large that the storage and direct inverse of $V_i = \mathrm{var}(Y_i|X_i)$ are impossible. Describe in detail how you will implement the EM algorithm to find the MLEs of $\beta$, $D$ and $\sigma^2$. Then apply your algorithm to a simulated data set which can be downloaded at the class website.