ST 755

Advanced Analysis of Variance

(Mixed Models and Variance Components)

Lecture Notes

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Contents

0 Preliminary

A: Matrix Algebra

A1: General results

1. For conformable matrices A, B, C, D,

$$\begin{vmatrix} A & 0 \\ C & D \end{vmatrix} = |A||D| \tag{0.1}$$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||D - CA^{-1}B| = |D||A - BD^{-1}C| \tag{0.2}$$

2. For any conformable P and Q, we have

$$|I + PQ| = |I + QP| \tag{0.3}$$

$$(I + PQ)^{-1} = I - P(I + QP)^{-1}Q \tag{0.4}$$

3. For any conformable matrices A, B, C, D,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix} \tag{0.5}$$

$$= \begin{bmatrix} A^{11} & -A^{11}BD^{-1} \\ -D^{-1}CA^{11} & D^{-1} + D^{-1}CA^{11}BD^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix} \tag{0.6}$$

$$= \begin{bmatrix} A^{-1} + A^{-1}BD^{11}CA^{-1} & -A^{-1}BD^{11} \\ -D^{11}CA^{-1} & D^{11} \end{bmatrix}$$

where $A^{11} = (A - BD^{-1}C)^{-1}$, $D^{11} = (D - CA^{-1}B)^{-1}$.

4. For any conformable matrices A and B,

$$\text{tr}(AB) = \text{tr}(BA).$$

5. If λ is an eigenvalue of a square matrix A (*i.e.*, $|\lambda I - A| = 0$), then $\exists$ a vector $x \neq 0$ such that

$$Ax = \lambda x.$$

6. If $A^2 = A$, then A is called idempotent. The eigenvalues of an idempotent matrix are either 0 or 1.

7. For a square matrix, define
$$\text{DET}(A) = \prod_{\lambda_i \neq 0} \lambda_i,$$
where λ_i's are the eigenvalues of A.

8. For $A_{m\times n}$ and $B_{n\times m}$, AB and BA have the same non-zero eigenvalues. Hence
$$\text{DET}(AB) = \text{DET}(BA).$$
Since $|\lambda I - AB| = \lambda^{m-n}|\lambda I - BA|$ (when $m \geq n$) if $\lambda \neq 0$.

A2: Symmetric matrices

1. If A is an m dimensional (real) symmetric matrix, then $\exists$ an orthonormal matrix H (i.e., $H^T H = HH^T = I$) such that
$$A = H\Lambda H^T,$$
where $\Lambda = diag\{\lambda_1, ..., \lambda_m\}$, λ_i's are the eigenvalues of A and the column vectors of H are the eigenvectors of A. Furthermore,
$$\text{tr}(A) = \sum \lambda_i, \quad |A| = \prod \lambda_i.$$

2. A symmetric matrix is said to be **positive definite** (non-negative or semi-positive definite) if all of its eigenvalues are positive (non-negative).

3. If A is a positive (non-negative) definite matrix, then $\exists$ a positive (non-negative) definite matrix B such that $A = B^2$. In this case, denote $B = A^{\frac{1}{2}}$.

4. If $A - B$ is non-negative definite for two non-negative definite matrices A, B, we say $A \geq B$.

5. For any matrix A, $AA^T \geq 0$.

A3: Matrix differentiation

1. If $A(\theta) = \{a_{ij}(\theta)\}$, then define
$$\frac{\partial A(\theta)}{\partial \theta_k} = \left\{\frac{\partial a_{ij}(\theta)}{\partial \theta_k}\right\}.$$

2. If A and x are conformable matrices, then

$$\frac{\partial(Ax)}{\partial x^T} = A. \tag{0.7}$$

3. If $A(\theta)$ and $B(\theta)$ are two conformable matrices, then

$$\frac{\partial\{A(\theta)B(\theta)\}}{\partial\theta_k} = \frac{\partial A(\theta)}{\partial\theta_k}\cdot B(\theta) + A(\theta)\cdot\frac{\partial B(\theta)}{\partial\theta_k}. \tag{0.8}$$

4. If $A(\theta)$ is nonsingular, then

$$\frac{\partial A^{-1}(\theta)}{\partial\theta_k} = -A^{-1}(\theta)\cdot\frac{\partial A(\theta)}{\partial\theta_k}\cdot A^{-1}(\theta). \tag{0.9}$$

5. If $|A(\theta)| \neq 0$, then

$$\frac{\partial|A(\theta)|}{\partial\theta_k} = |A(\theta)|\mathrm{tr}\left\{A^{-1}(\theta)\frac{\partial A(\theta)}{\partial\theta_k}\right\}. \tag{0.10}$$

6. If $|A(\theta)| > 0$, then

$$\frac{\partial\log(|A(\theta)|)}{\partial\theta_k} = \mathrm{tr}\left\{A^{-1}(\theta)\frac{\partial A(\theta)}{\partial\theta_k}\right\}. \tag{0.11}$$

$$\frac{\partial^2\log(|A(\theta)|)}{\partial\theta_j\partial\theta_k} = \mathrm{tr}\left\{A^{-1}(\theta)\frac{\partial^2 A(\theta)}{\partial\theta_j\partial\theta_k}\right\} - \mathrm{tr}\left\{A^{-1}(\theta)\frac{\partial A(\theta)}{\partial\theta_j}A^{-1}(\theta)\frac{\partial A(\theta)}{\partial\theta_k}\right\}. \tag{0.12}$$

7. Let P be the projection matrix $P = V^{-1} - V^{-1}X(X^TV^{-1}X)^{-1}X^TV^{-1}$ where X is a constant matrix. Then

$$\frac{\partial P}{\partial\theta} = -P\frac{\partial V}{\partial\theta}P.$$

B: Normal Random Variables

1. $X_{p\times 1} \sim N(\mu, \Sigma)$ $(\Sigma > 0)$ if X has density function

$$f(x;\mu,\Sigma) = (2\pi)^{-p/2}|\Sigma|^{-1/2}e^{-(x-\mu)^T\Sigma^{-1}(x-\mu)/2}, \tag{0.13}$$

implying

$$\int e^{-(x-\mu)^T\Sigma^{-1}(x-\mu)/2}dx = (2\pi)^{p/2}|\Sigma|^{1/2}. \tag{0.14}$$

2. If

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right),$$

then

$$X_1|X_2 = x_2 \sim N\{\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\}. \tag{0.15}$$

3. If $X_{p\times 1} \sim N(\mu, \Sigma)$ $(\Sigma \geq 0)$, then its moment generating function is

$$m(t) = \mathrm{E}(\mathrm{e}^{X^T t}) = \mathrm{e}^{t^T\mu + \frac{1}{2}t^T\Sigma t}.$$

C: Quadratic forms of normal random variables

1. Let $X \sim N(0, \sigma^2 I)$ and C is a symmetric matrix. Then

$$\frac{X^T C X}{\sigma^2} \sim \chi^2_r$$

$\Longleftrightarrow$ C is idempotent with rank r.

2. Let $X \sim N(0, \Sigma)$ and C is a symmetric matrix. Then

$$X^T C X \sim \chi^2_r$$

$\Longleftrightarrow$ $C\Sigma$ is idempotent with rank r (Since $Y = \Sigma^{-\frac{1}{2}}X \sim N(0, I)$).

3. Let $X \sim N(0, \Sigma)$ and C and D are symmetric matrices. Then

$$X^T C X \text{ and } X^T D X \text{ are independent}$$

$\Longleftrightarrow$ $C\Sigma D = 0$.

D: General Results for Random Variables

1. Let Y be a random variable with mean $\mathrm{E}(Y)$ and variance $V = \mathrm{Var}(Y)$ and A is a symmetric matrix. Then

$$\mathrm{E}(Y^T A Y) = \mathrm{E}(Y^T) A \mathrm{E}(Y) + \mathrm{tr}(AV).$$