Estimation and hypothesis testing in nonstationary time series

by

David Alan Dickey

An Abstract of

A Dissertation Submitted to the

Graduate Faculty in Partial Fulfillment of

The Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Approved:

[Signatures]

Iowa State University
Ames, Iowa

1976
Estimation and hypothesis testing in
nonstationary time series

David A. Dickey

Under the supervision of Wayne A. Fuller
From the Department of Statistics
Iowa State University

The autoregressive time series \( \{Y_t; t \in \{1, 2, \ldots\}\} \) where \( Y_t = \rho Y_{t-1} + e_t, \{e_t; t \in \{1, 2, \ldots\}\} \) is a sequence of normal independent (0, \( \sigma^2 \)) random variables and \( Y_0 = 0 \) is investigated. If \( |\rho| < 1 \) the process converges to a stationary time series as \( t \to \infty \). If \( |\rho| \geq 1 \) the process is called "explosive", and the variance is unbounded as \( t \to \infty \).

The maximum likelihood estimator of \( \rho \), \( \hat{\rho} = \left( \sum_{t=2}^{n} Y_{t-1}^2 \right)^{-1} \sum_{t=2}^{n} Y_t Y_{t-1} \) is studied. Also investigated are the regression estimators of \( \rho \) obtained when a constant is included in the regression of \( Y_t \) on \( Y_{t-1} \) and when a constant and a time trend are included in the regression. These two estimators are denoted by \( \hat{\rho}_\mu \) and \( \hat{\rho}_\tau \) respectively. The statistics constructed by analogy to the t-statistics for the regression coefficients of \( Y_{t-1} \) are denoted by \( \hat{r}, \hat{r}_\mu \) and \( \hat{r}_\tau \) for the regression of \( Y_t \) on \( Y_{t-1}, Y_t \) on (1), \( Y_{t-1} \) and \( Y_t \) on (1), \( t, Y_{t-1} \) respectively.

It is demonstrated that the distribution of \( \hat{\rho} \) given \( \rho = -1 \) is the mirror image of the distribution of \( \hat{\rho} \) given \( \rho = 1 \). It is also shown that the limiting distributions of \( \hat{\rho}_\mu \) and \( \hat{\rho}_\tau \) are the same as that of \( \hat{\rho} \) when \( \rho = -1 \).

The first and second moments for the numerators and denominators
of \( \hat{\rho}, \hat{\rho}_\mu \) and \( \hat{\rho}_\tau \) are derived. The numerator and denominator of \( \hat{\rho} \) are strongly correlated, but the numerators and denominators of \( \hat{\rho}_\mu \) and \( \hat{\rho}_\tau \) are asymptotically uncorrelated. The method of determining the moments may be of use in other calculations.

The eigenroots and eigenvectors for the numerator and denominator quadratic forms of \( n(\hat{\rho} - 1) \) and \( n(\hat{\rho}_\mu - 1) \) are derived. Two methods are used to obtain the limiting forms for the eigenroots and eigenvectors. The first method is a direct evaluation of the roots. The second technique involves showing that the limit of the normalized characteristic function associated with the length \( n \) time series converges to a limit function. The limiting distribution of \( n(\hat{\rho} - 1) \) is approximated by using the limit results for the eigenvectors and eigenroots.

Percentage points for the distribution of \( n(\hat{\rho} - 1), n(\hat{\rho}_\mu - 1) \) and \( n(\hat{\rho}_\tau - 1) \) are estimated using the Monte Carlo methods for \( n = 25, 50, 100, 250, 500, \) and \( \infty \) (\( \infty \) being the above mentioned limit case). Tables of these percentage points and percentage points for the distribution of \( \hat{\tau}, \hat{\tau}_\mu, \hat{\tau}_\tau \) are presented. The percentage points for \( \hat{\tau}, \hat{\tau}_\mu, \hat{\tau}_\tau \) differ substantially from those of Student's t. Extension theorems are presented which show that the distributional results hold if \( e_t \) is replaced by a stationary autoregressive time series.

The power of tests based on the above statistics is compared with the power of the Box and Jenkins Q statistic. Examples of tests using the statistics are given.
Estimation and hypothesis testing in nonstationary time series

by

David Alan Dickey

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of The Requirements for the Degree of DOCTOR OF PHILOSOPHY

Major: Statistics

Approved:

[Signature]
In Charge of Major Work

[Signature]
For the Major Department

[Signature]
For the Graduate College

Iowa State University
Ames, Iowa

1976
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>TITLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Statement of the Problem and Summary of Results</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Literature Review</td>
<td>2</td>
</tr>
<tr>
<td>II</td>
<td>ESTIMATORS</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Definitions of Estimators</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Properties of Estimators and Tests</td>
<td>8</td>
</tr>
<tr>
<td>III</td>
<td>MOMENTS OF NUMERATORS AND DENOMINATORS</td>
<td>11</td>
</tr>
<tr>
<td>IV</td>
<td>LIMIT DISTRIBUTIONS</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>Spectral Representation of $\sum_{t=2}^{n} Y_{t-1}^2$</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>Case I: $n(\delta - 1)$</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>Case II: $n(\hat{\delta}_\mu - 1)$</td>
<td>31</td>
</tr>
<tr>
<td></td>
<td>Case III: $n(\hat{\delta}_\tau - 1)$</td>
<td>34</td>
</tr>
<tr>
<td></td>
<td>Convergence in Distribution</td>
<td>36</td>
</tr>
<tr>
<td></td>
<td>Simulation</td>
<td>44</td>
</tr>
<tr>
<td>V</td>
<td>PERCENTILES OF THE DISTRIBUTIONS</td>
<td>49</td>
</tr>
<tr>
<td></td>
<td>Goodness of Fit</td>
<td>54</td>
</tr>
<tr>
<td></td>
<td>Standard Errors</td>
<td>55</td>
</tr>
<tr>
<td></td>
<td>Medians</td>
<td>56</td>
</tr>
<tr>
<td></td>
<td>Notes on Use of Tables</td>
<td>56</td>
</tr>
<tr>
<td></td>
<td>Distributions for Nonzero $\alpha$ and $\beta$</td>
<td>57</td>
</tr>
<tr>
<td></td>
<td>Comparisons with Other Procedures</td>
<td>76</td>
</tr>
<tr>
<td>VI</td>
<td>HIGHER ORDER PROCESSES</td>
<td>90</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>---------------------------------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>CHAPTER VII. EXAMPLES</td>
<td>107</td>
<td></td>
</tr>
<tr>
<td>Velocity of Money</td>
<td>107</td>
<td></td>
</tr>
<tr>
<td>Box and Jenkins Series D</td>
<td>109</td>
<td></td>
</tr>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>112</td>
<td></td>
</tr>
<tr>
<td>APPENDIX</td>
<td>113</td>
<td></td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>123</td>
<td></td>
</tr>
</tbody>
</table>
CHAPTER I. INTRODUCTION

Statement of the Problem and Summary of Results

We shall consider the first order autoregressive process

\[ Y_t = \rho Y_{t-1} + e_t \]  \hspace{1cm} (1.1)

where \( Y_0 = 0 \) and \( \{e_t\}_{t=1}^\infty \) is a sequence of normal independent \((0, \sigma^2)\) random variables. Notice that \( Y_t = e_t + \rho e_{t-1} + \cdots + \rho^{t-1} e_1 \) so that if \( |\rho| < 1 \) then \( Y_t \) converges to a stationary process as \( t \to \infty \) (in particular, note that \( \text{Var}(Y_t) \to \frac{\sigma^2}{1-\rho^2} \)). If \( |\rho| > 1 \) then \( \text{Var}(Y_t) \) grows without bound as \( t \to \infty \) and the process is called explosive. Given a realization \( Y_1, Y_2, \ldots, Y_n \) of a first order autoregressive time series, we shall be interested in estimators of \( \rho \) and in tests of the null hypothesis that \( \rho = 1 \).

Often, the alternative to the null hypothesis \( H_0: Y_t = Y_{t-1} + e_t \) is that the time series was generated according to the rule \( Y_t = \alpha + \rho Y_{t-1} + e_t \) where \( |\rho| < 1 \). A second alternative of interest is that the time series satisfies \( Y_t = \alpha + \beta t + \rho Y_{t-1} + e_t \) where \( |\rho| < 1 \). We shall discuss \( \hat{\rho} \), the maximum likelihood estimator for \( \rho \) in Model (1.1) and shall investigate the distribution of the estimator when \( |\rho| = 1 \). We also define and discuss two other estimators of \( \rho \) suggested by the two alternative models. In addition, \( t \)-type statistics are presented for the three estimators.

The estimators considered are quotients of quadratic forms in the \( e_t \). The first and second moments of these quadratic forms are presented. A representation useful in studying the limiting distribution of the
estimator $\hat{\rho}$ is given. Extensions of this representation for the other
estimators under consideration are presented.

Tables of percentiles of the distributions of our estimators for
$\rho = 1$ are generated by the Monte Carlo method. The accuracy of the
Monte Carlo results is evaluated.

The distribution of the maximum likelihood estimator $\hat{\rho}$ when $\rho = -1$
is shown to be a reflection of the distribution of the estimator with
$\rho = 1$. If $\rho = -1$, the limiting distributions for the other two esti-
mators of $\rho$ are equal to the limiting distribution of $\hat{\rho}$ when $\rho = -1$.

A theorem extending the results to higher order processes is given.
The use of these results is illustrated.

Literature Review

Although there is a large quantity of time series literature, most
of the results are for stationary time series. An early paper studying
the distribution of test statistics for time series is that of
Anderson (1942). In this article the author considers first and higher
order serial correlation coefficients for the Model (1.1) and develops
the distributions of these statistics for $\rho = 0$.

Another important paper studying the stationary autoregressive
process is that of Mann and Wald (1943). Let $\hat{\rho} = \left( \sum_{t=2}^{n} \frac{Y_{t-1}^2}{Y_t Y_{t-1}} \right)^{-1} \sum_{t=2}^{n} \frac{Y_t Y_{t-1}}{Y_t Y_{t-1}}$. Mann and Wald show that, for $|\rho| < 1$, $\hat{\rho}$ is the least squares estimator
and $n^{1/2}(\hat{\rho} - \rho)$ is asymptotically normally distributed with mean zero
and variance $1 - \rho^2$, given that the distribution of $e_t$ has finite
moments.
Rubin (1950) showed that under certain regularity conditions \( \hat{\beta} \) is a consistent estimator for all values of \( \rho \). White (1958, 1959) showed that, for \( |\rho| > 1 \) and \( e_t \text{ NID}(0, \sigma^2) \), the distribution of \( \hat{\rho} \) normalized by a function of \( n \) has a Cauchy distribution in the limit. His results can also be used to demonstrate that \((\hat{\beta} - 1)/\left(\sum_{t=2}^{n} e_t^2\right)^{1/2}\) (a statistic similar to the regression t-statistic) has a normal distribution in the limit. Although White was able to obtain the limit moment generating functions for the cases \( |\rho| < 1 \), \( |\rho| = 1 \), and \( |\rho| > 1 \), he was unable to invert the moment generating function for \( |\rho| = 1 \).

Reeves (1972) gives a method for evaluating points on the distribution function of \( \hat{\beta} \). An approximation to the distribution of a linear combination of chi-square variables is considered in some cases. Reeves gives a method that is exact in one case.

Let \( Y_t, t = 1, 2, \ldots \) be a collection of variables. The first difference operator \( \nabla \) is defined by \( \nabla Y_t = Y_t - Y_{t-1}, t = 2, 3, \ldots, n \). The \( d^{th} \) difference is defined recursively by \( \nabla^d Y_t = \nabla(\nabla^{d-1} Y_t) \) for \( d > 1 \).

Box and Jenkins (1970) introduce the term ARIMA (autoregressive integrated moving average) for a model in which the \( d^{th} \) difference is a stationary invertible autoregressive moving average time series.

Box and Jenkins argue that autoregressive moving average models having some of the characteristic roots of the autoregressive portion of modulus one are of great value in representing nonstationary time series that occur in practice (pg. 85). The analysis of such time series is accomplished by applying the methods for stationary invertible autoregressive moving averages to the time series of \( d^{th} \) differences.
(Box and Jenkins (1970) pg. 174). To determine if differencing is needed the autocorrelation function is inspected. If it dies out "slowly", differencing is suggested (pg. 175-177).

Once differencing has been accomplished and parameters estimated, Box and Jenkins suggest a $Q$ statistic to test the adequacy of the model. $Q$ is defined to be $n \sum_{k=1}^{K} r_k^2(\hat{\epsilon})$ where $r_k(\hat{\epsilon})$ is the $k^{th}$ serial correlation coefficient of the estimated residuals from the fitted model, $n$ is the number of observations available and $K$ is some "sufficiently large" constant. If the model is appropriate, $Q$ will be distributed approximately as a chi-square random variable.

Several authors have considered nonstationary processes in which one or more of the characteristic roots exceed one in absolute value while the rest are less than one in absolute value. Chow and Levitan (1969) use a method attributed to Quenouille (1957) to decompose $n$ observations on such a time series into an exponential trend and a detrended component. The spectral density of the detrended component is given in terms of the characteristic roots. This spectral density is shown to be of the same form as the spectral density of a stationary time series.

M. M. Rao (1961) studied autoregressive time series whose characteristic equations have a single root with absolute value exceeding one and remaining roots less than one in absolute value. He showed that under certain conditions the limiting distribution of the least squares estimate of the largest root is Cauchy or normal depending on the normalization used. Some consistency results are also given. A nice
feature of this paper is a table of known results concerning $\beta$.

Stigum (1974) also studied higher order autoregressive schemes with no roots of modulus 1. He defined a dynamic stochastic process to be the sum of a stationary process and a deterministic process. Limiting multivariate normal distributions (possibly singular) for the least squares autoregressive coefficients are obtained.

Tintner (1940) considers a model with a deterministic polynomial trend and independent errors. A method based on differences (called the variate difference method) is used to estimate the degree of the polynomial trend.

Quadratic forms involving the nxn matrix

$$B = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
0 & -1 & 2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 2
\end{pmatrix}$$

play an important role in our study. We show that the $i^{th}$ largest eigenvalue of $n^{-2}B^{-1}$ approaches $\frac{2}{(2i-1)n}$ as $n \to \infty$. Anderson (1971) studies quadratic forms similar to this one. In his Theorem 6.5.5 he shows that the characteristic roots of $A_1$ are $\cos \left( \frac{\pi s}{n+1} \right)$, $s = 1, 2, \ldots, n$ where $A_1$ is the nxn matrix.
\[ A_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \]

Since \(2(I-A_1)\) and \(B\) differ only in the top left element one wonders if the \(i^{th}\) largest eigenvalue of \(n^{-2}[2(I-A_1)]^{-1}\) also approaches \(\left(\frac{2}{(2i-1)\pi}\right)^2\) as \(n \to \infty\). This, however, is not the case. To see this, let \(\lambda\) be an eigenvalue of \(A_1\). Then \([2n^2(1-\lambda)]^{-1}\) is an eigenvalue of \([2n^2(I-A_1)]^{-1}\). Using Anderson's result, the \(i^{th}\) largest eigenvalue of \([2n^2(I-A_1)]^{-1}\) is

\[ [2n^2(1 - \cos(\frac{\pi i}{n+1}))]^{-1} \]

Now

\[
\lim_{n \to \infty} \frac{n^{-2}}{2(1 - \cos(\frac{\pi i}{n+1}))} =
\]

\[
\lim_{n \to \infty} \frac{n^{-3} \frac{(n+1)^3}{(\pi i)^2}}{\frac{(n+1)}{\pi i} \sin \left(\frac{\pi i}{(n+1)}\right)} = \left(\frac{1}{\pi i}\right)^2 .
\]
CHAPTER II. ESTIMATORS

Definitions of Estimators

Consider the model

\[ Y_t = \rho Y_{t-1} + e_t \]  \hspace{1cm} (2.0)

where \( Y_0 = 0 \) and \( \{e_t}\) is a sequence of NID(0, \( \sigma^2 \)) variables. We assume \( n \) observations \( Y_1, Y_2, \ldots, Y_n \) are available for analysis. Define the vectors \( l', t', Y_t, Y_{t-1} \) by

\[ l' = (1, 1, 1, \ldots, 1) \]  \hspace{1cm} (2.1)

\[ t' = (1 - \frac{n}{2}, 2 - \frac{n}{2}, 3 - \frac{n}{2}, \ldots, n - 1 - \frac{n}{2}) \]  \hspace{1cm} (2.2)

\[ Y'_t = (Y_2, Y_3, Y_4, \ldots, Y_n) \]  \hspace{1cm} (2.3)

\[ Y'_{t-1} = (Y_1, Y_2, Y_3, \ldots, Y_{n-1}) \]  \hspace{1cm} (2.4)

Define the regression coefficient obtained in the regression of

\[ Y_t \] on \( \tilde{l}_{t-1} \) by

\[ \hat{\beta} = (\sum_{t=2}^{n} Y_t Y'_{t-1})^{-1} \sum_{t=2}^{n} Y_t Y_{t-1}. \]  \hspace{1cm} (2.5)

Define \( \hat{\beta}_\mu \) to be the regression coefficient of \( Y_{t-1} \) in the regression of \( Y_t \) on \( l', \tilde{l}_{t-1} \) and define \( \hat{\beta}_\tau \) to be the regression coefficient of \( Y_{t-1} \) in the regression of \( Y_t \) on \( l', t', \tilde{l}_{t-1} \).

Define the statistics analogous to t-statistics by

\[ \tau = (\hat{\beta} - 1)/\sqrt{S^2_{e1}} \]  \hspace{1cm} (2.5)

\[ \tau_\mu = (\hat{\beta}_\mu - 1)/\sqrt{S^2_{e2}} \]  \hspace{1cm} (2.6)

\[ \tau_\tau = (\hat{\beta}_\tau - 1)/\sqrt{S^2_{e3}} \]  \hspace{1cm} (2.7)

where \( S^2_e \) is the appropriate regression residual mean square

\[ S^2_e = (n-k-1)^{-1}(Y'_{t-1}(I - \bar{X}(\bar{X}'\bar{X})^{-1}\bar{X}')Y_{t-1}). \]

\( \bar{X} \) is the \((n-1)xk\) matrix of independent regression variables from the
right hand side of the appropriate regression equation, and $c_i$ is the
element of $(X'X)^{-1}$ associated with $\hat{\beta}$, $\hat{\mu}$ and $\hat{\tau}$ respectively. Note that
the vector $\mathbf{y}_{t-1}$ is not independent of the errors in the regression so
that the usual distributional properties for regression estimators do not
hold.

Properties of Estimators and Tests

We shall obtain the likelihood function for $n$ observations generated
by Model (2.0). The vector $(e_2, e_3, \ldots, e_n)$ has probability density func-
tion

$$(2\pi \sigma^2)^{-\frac{n-1}{2}} \exp\left[-\frac{1}{2\sigma^2} \sum_{t=2}^{n} e_t^2 \right].$$

The transformation

$$
\begin{pmatrix}
e_2 \\
e_3 \\
e_4 \\
\vdots \\
e_n
\end{pmatrix} = 
\begin{pmatrix}
-\rho & 1 & 0 & \ldots & 0 \\
0 & -\rho & 1 & \ldots & 0 \\
0 & 0 & -\rho & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
\vdots \\
y_n
\end{pmatrix}
$$

(2.8)

has unit Jacobian and, thus, the inverse transformation also has unit
Jacobian. The logarithm of the likelihood function conditioned on $Y_1$ is

$$L(\rho | Y_1) = -\frac{n-1}{2} \log(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{t=2}^{n} (y_t - \rho y_{t-1})^2.$$  

(2.9)

The value of $\rho$ that maximizes (2.9) satisfies

$$\sum_{t=2}^{n} y_t y_{t-1} - \rho \sum_{t=2}^{n} y_t^2 = 0$$

and

$$\hat{\beta} = (\sum_{t=2}^{n} y_t^2)^{-1}(\sum_{t=2}^{n} y_t y_{t-1})$$
is the maximum likelihood estimator of $\rho$. This result is independent of the value of $\rho$.

Now consider the hypothesis $H_0: \rho = 1$ and the three alternatives $H_A: \rho \in C_i$ where $C_1 = (-\infty, 1)$, $C_2 = (1, \infty)$, and $C_3 = (-\infty, 1) \cup (1, \infty)$. We shall show that the likelihood ratio tests for these hypotheses are based on the statistic $\tau$. Under $H_0$, the maximum of the likelihood function is

$$\left[2\pi(n-1)^{-1} \sum_{t=2}^{n} (Y_t - Y_{t-1})^2\right]^{-\frac{(n-1)}{2}} \exp\left(-\frac{(n-1)}{2}\right). \quad (2.10)$$

Under the alternative the maximum likelihood estimator of $\sigma^2$ is

$$\frac{1}{n-1} \sum_{t=2}^{n} (Y_t - \hat{\rho}_M Y_{t-1})^2$$

where $\hat{\rho}_M$ is the maximum likelihood estimator of $\rho$.

If $\hat{\rho}_M = \hat{\rho}$ the maximum of the likelihood function is

$$\left[2\pi(n-1)^{-1} \sum_{t=2}^{n} (Y_t - \hat{\rho} Y_{t-1})^2\right]^{-\frac{(n-1)}{2}} \exp\left(-\frac{(n-1)}{2}\right). \quad (2.11)$$

Suppose $C_i \neq C_3$ and $\hat{\rho} \in C_i$. For any fixed value of $\sigma^2$, the log likelihood function is a parabola in $\rho$ opening down. Therefore the supremum over $C_i$ (i $\neq$ 3) of the likelihood function is found by maximizing the function over the set $\{(\rho, \sigma^2): \rho = 1\}$. This shows that the likelihood ratio is 1 if $\hat{\rho} \in C_i$. That is $H_0$ is accepted if $H_A$ is $C_1$ and $\hat{\rho} > 1$ and $H_0$ is accepted if $H_A$ is $C_2$ and $\hat{\rho} < 1$.

For $\hat{\rho} \in C_i$ the likelihood ratio test rejects $H_0$ when

$$\sum_{t=2}^{n} (Y_t - Y_{t-1})^2 / \sum_{t=2}^{n} (Y_t - \hat{\rho} Y_{t-1})^2$$

is large. Now $\sum_{t=2}^{n} (Y_t - Y_{t-1})^2 = \sum_{t=2}^{n} (Y_t - \hat{\rho} Y_{t-1})^2 + (1 - \hat{\rho})^2 \sum_{t=2}^{n} Y_{t-1}^2$ so the likelihood ratio test rejects

$$\frac{(1-\hat{\rho})^2 \sum_{t=2}^{n} Y_{t-1}^2}{\sum_{t=2}^{n} (Y_t - \hat{\rho} Y_{t-1})^2} > a$$

when $1 + \frac{\sum_{t=2}^{n} (Y_t - \hat{\rho} Y_{t-1})^2}{\sum_{t=2}^{n} Y_{t-1}^2} > a$ where a is some constant greater than 1.
Thus $H_0$ is rejected when

$$\frac{\left| 1 - \hat{\rho} \right|}{\sqrt{\sum_{t=2}^{n} (Y_t - \hat{\rho} Y_{t-1})^2}} > \sqrt{(a-1)(n-2)}.$$  \hspace{1cm} (2.12)

The expression on the left in (2.12) is $|\tau|$. Thus, the likelihood ratio test accepts $H_0$ if $\hat{\rho} \notin C_1$ or if $|\tau| \leq \sqrt{(a-1)(n-2)}$. Otherwise $H_0$ is rejected. Note that the outlined procedure is the ordinary one-sided test for $H_A: \rho \in C_1$ or $H_A: \rho \in C_2$. 
CHAPTER III. MOMENTS OF NUMERATORS AND DENOMINATORS

In this section we obtain the first two moments of the numerator and denominator quadratic forms appearing in the estimators of \( \hat{\rho} \). The method, based on elementary properties of polynomials, results in considerable saving of effort over straight calculation, particularly for \( \hat{\rho}_T \).

Using \( \hat{\rho}_T \) as the first example we note that \( \hat{\rho}_T \) can be expressed as

\[
\hat{\rho}_T = \frac{Y_t' (I - M') \tilde{X}_T - 1}{\tilde{X}_T - 1 \tilde{M'} \tilde{Y}_T - 1}
\]

(3.1)

where

\[
\tilde{X} = \begin{bmatrix}
1 & 1 - \frac{n}{2} \\
1 & 2 - \frac{n}{2} \\
1 & 3 - \frac{n}{2} \\
& \\
& \\
& \\
1 & n - 1 - \frac{n}{2}
\end{bmatrix}, \quad \tilde{M'} = \tilde{X}(\tilde{X}'\tilde{X})^{-1}\tilde{X}',
\]

\( \tilde{e}_t = (e_1, e_2, e_3, \ldots, e_n) \), \( \tilde{Y}_T = B\tilde{C}e_t \), \( \tilde{Y}_T = A\tilde{C}e_t = \tilde{Y}_T - 1 + A\tilde{e}_t \),

\( A = (\tilde{e}_n \tilde{e}_n) \), \( B = (\tilde{I}_n \tilde{I}_n) \) and

\[
\tilde{C} = \begin{bmatrix}
1 & 0 & 0 & 0 & \ldots & 0 \\
1 & 1 & 0 & 0 & \ldots & 0 \\
1 & 1 & 1 & 0 & \ldots & 0 \\
& & & & \ddots & \vdots \\
& & & & & \ddots \\
& & & & & & \ddots \\
1 & 1 & 1 & 1 & \ldots & 1
\end{bmatrix}
\]

Here \( \tilde{e}_n \) is a length \( n-1 \) vector of zeros. Thus
the expression for general n.

B. Expectation of the Denominator of $\hat{\beta}_1$:

The expectation is

$$\sigma^2 \text{tr}(C'B'DBC).$$

Now $(C'B'DBC)_{ij} = \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} d_{jk}$ and the expectation for the denominator is given by

$$\sigma^2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} d_{jk} = \frac{\sigma^2}{n(n-1)(n-2)} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} (n(n-1)(n-2) \delta_{jk} - n(n-2))$$

$$- 12 \left( \frac{n}{2} - \frac{n}{2} \right) (k - \frac{n}{2}).$$

The triple sum is a fifth degree polynomial in n which can be completely determined by finding its value at 6 different values of n.

C. Expectation of the square of the numerator of $\hat{\beta}_1$:

We note that for a vector $\varepsilon_i$ of uncorrelated $(0, \sigma^2)$ random variables and a square matrix of constants $F$

$$E(\varepsilon_i^T F \varepsilon_i)^2 = E(\varepsilon_i^T \sum_{j=1}^{n} \sum_{i'=1}^{n} \sum_{j'=1}^{n} e_i e_{i'} e_{j'} f_{ij} f_{i'j'} i') =$$

$$\sigma^4 \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ f_{ij} f_{ji} + f_{ii} f_{jj} + f_{ij} f_{ij} \right].$$

We have seen that $f_{ij}$ for the numerator is $\frac{\sigma^2}{n(n-1)(n-2)}$ times a polynomial in $n^3$, $n^2 j$, $n j^2$, and $i n j$. Thus $f_{ij}^2 f_{ji}$ involves a sixth degree polynomial in $n$, $i$, and $j$, and $\sigma^4 \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ f_{ij} f_{ji} + f_{ii} f_{jj} + f_{ij} f_{ij} \right]$ is $\frac{\sigma^4}{n^2(n-1)^2(n-2)^2}$ times an eighth degree polynomial in $n$.

D. Expectation of the square of the denominator of $\hat{\beta}_1$:

Using (3.4) we see that the expected value of the square of the denominator is a tenth degree polynomial in $n$ multiplied by $\frac{\sigma^4}{n^2(n-1)^2(n-2)^2}$.

E. Expectation of the numerator times the denominator of $\hat{\beta}_1$:

Letting $F(N)$ and $F(D)$ represent the matrices in the numerator and denominator
quadratic forms, respectively, and letting $f_{ij}^{(N)}$ and $f_{ij}^{(D)}$ be the $ij^{th}$ elements of these matrices we find

$$E(g_t' F^{(N)} g_t g_t' F^{(D)} g_t) = \sigma^2 \sum_{i=1}^{n} \sum_{j=1}^{n} [f_{ij}^{(N)} f_{ji}^{(D)} + f_{ii}^{(N)} f_{jj}^{(D)} + f_{ij}^{(N)} f_{ij}^{(D)}].$$

This is $\frac{\sigma^2}{n^2(n-1)^2(n-2)}$ times a ninth degree polynomial in $n$.

The IBM 360 computer was used to generate the sums of powers needed in the calculations of the polynomial coefficients. These values are all integers and no round-off error occurred. An $11 \times 11$ matrix $G$ was also created with $ij^{th}$ element $g_{ij} = (i + 3)^{j-1}$ so that if $\tilde{b}' = (b_0, b_1, b_2, \ldots, b_{10})$ is the vector of coefficients in the polynomial $b_0 + b_1 n + b_2 n^2 + \ldots + b_{10} n^{10}$ then setting $G \tilde{b}$ equal to the appropriate vector of sums will yield the polynomials needed in the expectations previously discussed.

Before solving each system, elementary row operations were performed to reduce the system to $G^* w_i = \tilde{x}_i$, $i = A, B, \ldots, E$ where $G^*$ is an upper triangular matrix and $\tilde{x}_i$ is the corresponding transformed vector of sums of case $i$. Having obtained this form the system of equations was solved by hand so that, again, the coefficients were computed with no round-off error.

Display (3.5) below gives the matrix $G^*$ used in the calculations.
<table>
<thead>
<tr>
<th>1</th>
<th>4</th>
<th>16</th>
<th>64</th>
<th>256</th>
<th>1,024</th>
<th>65,536</th>
<th>292,144</th>
<th>1,048,576</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>61</td>
<td>369</td>
<td>2,101</td>
<td>11,529</td>
<td>61,741</td>
<td>325,089</td>
<td>1,690,981</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>302</td>
<td>2,550</td>
<td>19,502</td>
<td>140,070</td>
<td>963,902</td>
<td>6,433,590</td>
<td>41,983,502</td>
</tr>
<tr>
<td>6</td>
<td>132</td>
<td>1,830</td>
<td>20,460</td>
<td>201,726</td>
<td>1,832,292</td>
<td>15,717,750</td>
<td>129,325,020</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>720</td>
<td>13,080</td>
<td>186,480</td>
<td>3,298,744</td>
<td>25,719,120</td>
<td>268,623,960</td>
<td></td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>4,680</td>
<td>107,520</td>
<td>1,900,080</td>
<td>28,594,440</td>
<td>385,945,460</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>720</td>
<td>35,280</td>
<td>997,920</td>
<td>21,379,680</td>
<td>385,363,440</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5,040</td>
<td>302,400</td>
<td>10,311,840</td>
<td>263,088,000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40,320</td>
<td>2,903,040</td>
<td>117,331,220</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>362,880</td>
<td>30,844,800</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3,628,800</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 3.1. Display (3.5)
In Table 3.1 are the vectors $\xi_d$.

Table 3.1. Vectors for $\xi_d$.

<table>
<thead>
<tr>
<th>i</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>-12</td>
<td>8</td>
<td>480</td>
<td>192</td>
<td>-</td>
<td>288</td>
</tr>
<tr>
<td>-48</td>
<td>40</td>
<td>7,440</td>
<td>4,560</td>
<td>-</td>
<td>5,472</td>
</tr>
<tr>
<td>-72</td>
<td>80</td>
<td>4,360</td>
<td>40,800</td>
<td>-</td>
<td>39,168</td>
</tr>
<tr>
<td>-48</td>
<td>80</td>
<td>134,280</td>
<td>193,680</td>
<td>-148,032</td>
<td></td>
</tr>
<tr>
<td>-12</td>
<td>40</td>
<td>2,3072</td>
<td>558,720</td>
<td>-336,096</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>8</td>
<td>269,568</td>
<td>1,043,856</td>
<td>-484,128</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>180,672</td>
<td>1,295,520</td>
<td>-447,552</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>67,358</td>
<td>1,053,920</td>
<td>-258,048</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>10,752</td>
<td>556,800</td>
<td>-84,672</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>168,480</td>
<td>-12,069</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>22,464</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Solving the systems the following polynomials are obtained:

\[
E(\text{Numerator}) = -\frac{\sigma^2}{2}(n-3),
\]

\[
E(\text{Denominator}) = \frac{\sigma^2}{15}(n+1)(n-3),
\]

\[
E(\text{Numerator}^2) = \frac{\sigma^4}{30}(n-3)(8n-7),
\]

\[
E(\text{Denominator}^2) = \frac{\sigma^4}{2100}(n-3)(n+1)(13n^2-26n+36), and
\]

\[
E(\text{Numerator} \cdot \text{Denominator}) = -\frac{\sigma^4}{30}(n+1)(n-3)(n-1).
\]

This method can also be applied to the estimator $\hat{\rho}_\mu$ by using a column of 1's as the X matrix. Thus in (3.2) we let $M_{\hat{\mu}}_{ij} = \frac{1}{n-1}$. Now, for example, we could write out the expectation of the numerator as
\[ \sigma^2 \sum_{i=2}^{n-1} \sum_{k=1}^{n-1} d_{i-1,k} = \frac{2}{n-1} \sum_{i=2}^{n-1} \sum_{k=1}^{n-1} (-1) \]

which is \( \frac{\sigma^2}{n-1} \) times a second degree polynomial in \( n \). The expectation of the denominator is

\[ \sigma^2 \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} d_{\ell=1,k} = \frac{2}{n-1} \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} ((n-1) \delta_{\ell,k} - 1) \]

which is \( \frac{\sigma^2}{n-1} \) times a third degree polynomial in \( n \).

By the same reasoning \( \mathbb{E}(\text{Numerator}^2) \) is \( \frac{\mu}{(n-1)^2} \) times a fourth degree polynomial, \( \mathbb{E}(\text{Numerator} \cdot \text{Denominator}) \) is \( \frac{\mu}{(n-1)^2} \) times a fifth degree polynomial and \( \mathbb{E}(\text{Denominator}^2) \) is \( \frac{\mu}{(n-1)^2} \) times a sixth degree polynomial. To obtain the coefficients we solve \( G^{\star} b_{\ell,i} = \gamma_{\ell,i} \) for the \( b_{\ell,i} \) where \( G^{\star} \) is the upper left 7 x 7 corner of the \( G^{\star} \) matrix used for \( \beta \) and the \( \gamma_{\ell,i} \) columns are given in Table 3.2.

<table>
<thead>
<tr>
<th>i</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>4</td>
<td>24</td>
<td>36</td>
<td>-24</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>6</td>
<td>56</td>
<td>168</td>
<td>-76</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
<td>64</td>
<td>408</td>
<td>-124</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>36</td>
<td>567</td>
<td>-111</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>8</td>
<td>456</td>
<td>-52</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>198</td>
<td>-10</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>36</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

The following are the moments of the numerator and denominator of
\( \hat{\beta}_\mu : \)

\[
E(\text{Numerator}) = -\frac{\sigma^2}{2}(n-2),
\]

\[
E(\text{Denominator}) = \frac{\sigma^4}{20}n(n-2),
\]

\[
E(\text{Numerator}^2) = \frac{\sigma^4}{3}n(n-2),
\]

\[
E(\text{Denominator}^2) = \frac{\sigma^4}{20}(n-2)(n^2-2n+2)(n), \text{ and}
\]

\[
E(\text{Numerator} \cdot \text{Denominator}) = -\frac{\sigma^4}{12}n^2(n-2).
\]

To obtain the corresponding results for \( \hat{\beta} \) we let \( \mathbf{M}_\text{X} \) be the zero matrix and find that \( E(\text{Numerator}) = 0 \), \( E(\text{Denominator}) \) is a second degree polynomial, \( E(\text{Numerator}^2) \) is a second degree polynomial, \( E(\text{Denominator}^2) \) is a fourth degree polynomial and \( E(\text{Numerator} \cdot \text{Denominator}) \) is a third degree polynomial.

We now use \( G_{2,21}^* = \Xi_{21} \) where \( G_2^* \) is the upper left 5 x 5 submatrix of the original \( G^* \) and the \( \Xi_{21} \) are listed in Table 3.3.

**Table 3.3. Vectors for \( \hat{\beta} \).**

<table>
<thead>
<tr>
<th>i</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
<td>6</td>
<td>88</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>4</td>
<td>152</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>143</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>70</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>14</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

The moments of the numerator and denominator of \( \hat{\beta} \) are:
E(Numerator) = 0,
E(Denominator) = \frac{\sigma^2}{2} n(n-1),
E(Numerator^2) = \frac{\sigma^4}{2} n(n-1),
E(Denominator^2) = \frac{\sigma^4}{12} n(n-1)(7n^2 - 7n + 1), and
E(Numerator \cdot Denominator) = \frac{\sigma^4}{3} n(n-1)(n-2).

In the case of \hat{\beta} it is as easy to find these expressions directly by expressing the numerator and denominator in terms of the e_t's.

Using the expectations obtained above, we get the moments displayed in Table 3.4. Recall that (n-1) squares appear in the denominators of the estimators.

Table 3.4. Moments of numerators and denominators.

<table>
<thead>
<tr>
<th>Statistics</th>
<th>\hat{\beta}</th>
<th>\hat{\beta}_\mu</th>
<th>\hat{\beta}_\tau</th>
</tr>
</thead>
<tbody>
<tr>
<td>EN</td>
<td>0</td>
<td>- \frac{\sigma^2}{2} (n-2)</td>
<td>- \frac{\sigma^2}{2} (n-3)</td>
</tr>
<tr>
<td>VN</td>
<td>\frac{\sigma^4}{2} n(n-1)</td>
<td>\frac{\sigma^4}{12} (n-2)(n+6)</td>
<td>\frac{\sigma^4}{60} (n-1)(n+31)</td>
</tr>
<tr>
<td>ED</td>
<td>\frac{\sigma^2}{2} n(n-1)</td>
<td>\frac{\sigma^2}{6} n(n-2)</td>
<td>\frac{\sigma^2}{15} (n+1)(n-3)</td>
</tr>
<tr>
<td>VD</td>
<td>\frac{\sigma^4}{3} n(n-1)(n^2-n+1)</td>
<td>\frac{\sigma^4}{90} n(n-2)(2n^2-4n+9)</td>
<td>\frac{\sigma^4}{6300} (n+1)(n-3)(11n^2-22n+192)</td>
</tr>
<tr>
<td>CV</td>
<td>\frac{\sigma^4}{3} n(n-1)(n-2)</td>
<td>- \frac{\sigma^4}{6} n(n-2)</td>
<td>- \frac{\sigma^4}{15} (n+1)(n-3)</td>
</tr>
</tbody>
</table>
In Table 3, EN and ED are expectations of the numerators and denominators while VN and WD are their variances. CV is the covariance.

It is easily seen that the correlation of the numerator and denominator approaches 0 as \( n \to \infty \) in the \( \hat{\mu} \) and \( \hat{\nu} \) cases. In the \( \hat{\rho} \) case the correlation approaches .8165.
CHAPTER IV. LIMIT DISTRIBUTIONS

Spectral Representation of \( \sum_{t=2}^{n} \gamma_{t-1} \)

We shall obtain the spectral decomposition associated with

\[
\lim_{n \to \infty} n^{-2} \sum_{t=2}^{n} \gamma_{t-1}.
\]

This decomposition is characterized by a few dominant eigenvalues. Two methods for finding the limit spectral decomposition are given. The first finds an expression for the \( i \)th largest eigenvalue associated with \( n^{-2} \sum_{t=2}^{n} \gamma_{t-1} \) and then takes the limit. This method depends on a result of Rutherford (1946) and gives the spectral representation of the denominator of \( \hat{\rho} \) for finite \( n \). A second method (due to this author and developed prior to his knowledge of the Rutherford result) is presented in an appendix. This method was used to obtain the limiting representations for the quadratic forms in the denominators of \( \hat{\rho} \) and \( \hat{\mu} \). In addition the limiting characteristic equations of the matrices appearing in the quadratic forms are obtained.

The Rutherford result states that the determinant of the \( mxm \) matrix

\[
R_m(x,a,b) = \begin{bmatrix}
x+b & 1 & 0 & 0 & 0 \\
1 & x & 1 & \cdots & 0 & 0 \\
0 & 1 & x & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & x & 1 \\
0 & 0 & 0 & \cdots & 1 & x+a
\end{bmatrix}
\]

is given by

\[
|R_m(x,a,b)| = \frac{\sin(m+1)\theta + (a+b)\sin(m\theta) + ab \sin(m-1)\theta}{\sin \theta}
\]

where \( x = 2 \cos \theta \).
We note that the quadratic form \( \sum_{t=2}^{n} y_{t-1}^2 \) expressed in terms of \( e'_{n} = (e'_1, e'_2, \ldots, e'_{n-1}) \) is
\[
\begin{pmatrix}
  n-1 & n-2 & n-3 & \ldots & 1 \\
  n-2 & n-2 & n-3 & \ldots & 1 \\
  n-3 & n-3 & n-3 & \ldots & 1 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & 1 & 1 & \ldots & 1
\end{pmatrix}
\]
\[e'_{n} = e'_{n} A_{n} e'_{n}. \quad (4.1)\]

If no root of the matrix \( A_{n} \) is 0, then the roots are also given by the roots of \( |A_{n}^{-1} + \lambda^{-1}I| = 0 \). Now
\[
A_{n}^{-1} = \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & \ldots & 0 \\
0 & 1 & -2 & 1 & 0 \\
0 & 0 & 1 & -2 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & -2
\end{bmatrix}
\]

Therefore \( |A_{n}^{-1} + \lambda^{-1}I| = |R_{n-1}(-2 + \lambda^{-1}, 0, 1)| \). Setting
\[
|R_{n-1}(-2 + \lambda^{-1}, 0, 1)| = 0 \quad \text{we have}
\]
\[
\frac{\sin(n\theta) + \sin((n-1)\theta)}{\sin \theta} = 0.
\]

Equality occurs whenever \( \frac{n\theta + (n-1)\theta}{2} = k\pi \), that is whenever \( \theta = 2\pi k = \frac{2k\pi}{2n-1} \) for \( k = 1, 2, \ldots, n-1 \). Let \( \lambda_{i, n} \) be the eigenvalue of \( A_{n} \) associated with \( 2\theta(n-i) = 2\left(\frac{(n-i)\pi}{2n-1}\right) \), \( i = 1, 2, \ldots, n-1 \). By Rutherford's result,
\[
\lambda_{i, n}^{-1} = 2 + 2 \cos(2\theta(n-i)) = 2[1 + \cos^2\theta(n-i) - \sin^2\theta(n-i)]
\]
\[
= 2[\cos^2\theta(n-i) + (1 - \sin^2\theta(n-i))] = 1 + \cos^2\theta(n-i).
\]
Thus

\[ \lambda_{i,n} = \frac{1}{4} \sec^2 \theta(n-1) = \frac{1}{4} \sec^2 \left( \frac{(n-1)\pi}{2n-1} \right), \quad i = 1, 2, \ldots, n-1. \quad (4.2) \]

As a check of this result, the eigenvalues of \( A_{26} \) (a 25x25 matrix) were calculated using the IMSL subroutine EIGRF. The largest five of the roots obtained by that program are compared to \( \frac{1}{4} \sec^2 \left( \frac{(26-i)\pi}{51} \right) \), \( i = 1, 2, \ldots, 5 \) in Table 4.1. Calculations were carried out in double precision.

Table 4.1. Largest five eigenvalues of \( A_{26} \) computed by alternative methods.

<table>
<thead>
<tr>
<th>IMSL</th>
<th>263.617</th>
<th>29.365</th>
<th>10.625</th>
<th>5.462</th>
<th>3.338</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{4} \sec^2 \theta(n-1) )</td>
<td>263.614</td>
<td>29.365</td>
<td>10.625</td>
<td>5.462</td>
<td>3.338</td>
</tr>
</tbody>
</table>

Now we wish to find the limit of the \( i \)th largest normalized eigenvalue \( \lambda_{i,n} \) as \( n \rightarrow \infty \). The proper normalizing constant is \( n^2 \). This normalization is suggested because the sum of the eigenvalues is \( \text{tr} \Lambda_n = \frac{n(n-1)}{2} \). The normalization is verified by showing that the \( i \)th largest eigenvalue divided by \( n^2 \) approaches a finite nonzero limit as \( n \rightarrow \infty \).

We have

\[
\lim_{n \rightarrow \infty} n^{-1/2} \lambda_{i,n} = \lim_{n \rightarrow \infty} \left( 2n^{-1} \right) \sec \left( \frac{(n-1)\pi}{2n-1} \right) = \lim_{n \rightarrow \infty} \frac{n^{-1}}{2} \cos \left( \frac{(n-1)\pi}{2n-1} \right)
\]

\[
= \lim_{n \rightarrow \infty} \frac{n^{-2}}{2 \left[ \sin \left( \frac{(n-1)\pi}{2n-1} \right) \right] \left( \frac{(2n-1)}{2n-1} \right)^{2}} = \frac{2}{(2i-1)\pi} \quad (4.3)
\]

and thus

\[
\lim_{n \rightarrow \infty} n^{-2} \lambda_{i,n} = \left( \frac{2}{(2i-1)\pi} \right)^2. \quad (4.4)
\]
Note that if \( k \) is constant (not a function of \( n \) ) then
\[
\lim_{n \to \infty} (2n)^{-1} \sec \left( \frac{kn}{2n-1} \right) = 0.
\]

A few normalized eigenvalues for \( A_{26} \) are given in Table 4.2 along with the corresponding values of \( h((2i-1)n)^{-2} \).

<table>
<thead>
<tr>
<th>IMSEL</th>
<th>( \frac{2}{(2i-1)n} )</th>
<th>( \frac{2}{(2i-1)n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.38997</td>
<td>.40529</td>
<td>.04344</td>
</tr>
</tbody>
</table>

We now obtain the eigenvector of \( A_n \) corresponding to \( \lambda_{i,n} \) where \( A_n \) is defined in (4.1) and \( \lambda_{i,n} \) in (4.2). Let \( \vec{x}_{i,n} = (x_1, x_2, x_3, \ldots, x_{n-1}) \) be the eigenvector and define operators \( \Delta \) and \( \Delta_\perp \) as follows: for any matrix (vector) \( B \) with \( n \) rows \( \Delta B \) is a matrix (vector) of the same dimension as \( B \), having the same first row as \( B \) and having \( i^{th} \) \((i > 1)\) row equal to the \( i^{th} \) row of \( B \) minus the \((i-1)^{st}\) row of \( B \). Let \( \Delta_\perp B \) be a matrix of the same dimension as \( B \) with the \( n^{th} \) row of \( \Delta_\perp B \) equal to the \( n^{th} \) row of \( B \) and the \( i^{th} \) \((i < n)\) row of \( \Delta_\perp B \) equal to the \( i^{th} \) row of \( B \) minus the \((i+1)^{st}\) row of \( B \). Note that \( \Delta(\Delta_\perp A_n) = I \). The characteristic equation
\[
A_n \vec{x}_{i,n} = \lambda_{i,n} \vec{x}_{i,n} \quad (4.5)
\]
is equivalent to
\[
\Delta(\Delta_\perp A_n) \vec{x}_{i,n} = \lambda_{i,n} \Delta(\Delta_\perp \vec{x}_{i,n}),
\]
that is,
\[
\vec{x}_{i,n} = \lambda_{i,n} \Delta(\Delta_\perp \vec{x}_{i,n}).
\]
This system is

\[ x_1 = -\lambda_i n (x_2 - x_1) \]
\[ x_2 = -\lambda_i n (x_3 - 2x_2 + x_1) \]
\[ x_3 = -\lambda_i n (x_4 - 2x_3 + x_2) \]
\[ \vdots \]

\[ x_{n-1} = -\lambda_i n (-2x_{n-1} + x_{n-2}). \]

(Recall that \( A_n \) is an \((n-1)x(n-1)\) matrix.)

We normalize the eigenvector by setting \( x_1 = 1 \). Then \( x_2 = 1 - \lambda_i^{-1} \).

Using \( x_1 \) and \( x_2 \) as initial conditions, the difference equation

\[ X_t - (2 - \lambda_i^{-1})X_{t-1} + X_{t-2} = 0 \]

can be solved by standard techniques (see for example Fuller (1976)).

The solution to the difference equation is

\[ X_t = r^t (d_1 \cos \theta t + d_2 \sin \theta t) \]

where

\[ r = 1, \sin \theta = \frac{\sqrt{4\lambda_i n - 1}}{2\lambda_i n}, \cos \theta = 1 - (2\lambda_i n)^{-1}. \]

By the initial conditions, \( d_1 \cos \theta + d_2 \sin \theta = 1 \) and \( d_1 \cos(2\theta) + d_2 \sin(2\theta) = 1 - (\lambda_i n)^{-1} \). It follows that \( d_1 = 1 \) and \( d_2 = \left(4\lambda_i n - 1\right)^{-1/2} \).

To summarize, the eigenvector \( X_{4,n} \) associated with \( \lambda_i n \) has \( t \)th element given by \( x_t,1 \) where

\[ x_t,1 = \cos(\theta_t t) + \left(4\lambda_i n - 1\right)^{-1/2} \sin(\theta_t t), \ t=1,2,\ldots,n-1 \] (4.6)

and

\[ \theta_t = \arccos[1 - (2\lambda_i n)^{-1}]. \] (4.7)
We now investigate \( \lim_{n \to \infty} \frac{1}{n-1} \sum_{t=1}^{n-1} x_{t,i}^2 \). We have
\[
\frac{1}{n-1} \sum_{t=1}^{n-1} x_{t,i}^2 = \frac{1}{n-1} \sum_{t=1}^{n-1} \left[ \cos(\theta_1 t) + (4\lambda_{i,n} - 1)^{-\frac{1}{2}} \sin(\theta_1 t) \right]^2.
\] (4.8)

Since, by (4.3),
\[
\lim_{n \to \infty} \frac{\lambda_{i,n}}{n^2} = \left(\frac{2}{(2i-1)^2}\right),
\]
it follows that for fixed \( i \)
\[
\frac{\sqrt{4\lambda_{i,n} - 1}}{2\lambda_{i,n}} = o(\frac{1}{n}).
\]

By Taylor's series we obtain
\[
\theta_i = \arcsin\left(\frac{\sqrt{4\lambda_{i,n} - 1}}{2\lambda_{i,n}}\right) = \frac{\sqrt{4\lambda_{i,n} - 1}}{2\lambda_{i,n}} + O(n^{-3}).
\]

Furthermore
\[
(4\lambda_{i,n} - 1)^{-\frac{1}{2}} \sin(\theta_1 t) = O(n^{-1})
\] (4.8a)

because the sine function is bounded. Using these facts we will show that
\[
\lim_{n \to \infty} \frac{1}{n-1} \sum_{t=1}^{n-1} x_{t,i}^2 = \frac{1}{2}.
\]

Using (4.5),
\[
n^{-1} \sum_{t=1}^{n-1} x_{t,i}^2 = n^{-1} \sum_{t=1}^{n-1} \left[ \cos^2(\theta_1 t) + 2(4\lambda_{i,n} - 1)^{-\frac{1}{2}} \cos(\theta_1 t) \sin(\theta_1 t)
\right.
\]
\[
+ (4\lambda_{i,n} - 1)^{-1} \sin^2(\theta_1 t) \right] = n^{-1} \sum_{t=1}^{n-1} \cos^2(\theta_1 t)
\]
\[
+ n^{-1} \sum_{t=1}^{n-1} (4\lambda_{i,n} - 1)^{-1/2} \sin(2\theta_1 t) + O(n^{-2}),
\] (4.9)

and we used \( 2 \cos \alpha \sin \alpha = \sin(2\alpha) \). By (4.8a) we can write (4.9) as
\[
n^{-1} \sum_{t=1}^{n-1} x_{t,i}^2 = n^{-1} \sum_{t=1}^{n-1} \cos^2(\theta_1 t) + O(n^{-1}).
\]
Using Jolley (1961, pg. 82 no. 4.38) we obtain

$$\lim_{n \to \infty} n^{-1} \sum_{t=1}^{n-1} \cos^2(\theta_{t,1}) = \lim_{n \to \infty} n^{-1} \left[ \frac{n-1}{2} + \frac{\cos(n\theta_{1}) \sin((n-1)\theta_{1})}{2 \sin \theta_{1}} \right]. \quad (4.10)$$

Now

$$\lim_{n \to \infty} n \theta_{1} = \frac{(2i-1)\pi}{2}, \quad (4.11)$$

$$\lim_{n \to \infty} \cos(n\theta_{1}) \sin((n-1)\theta_{1}) = \cos(\frac{(2i-1)\pi}{2}) \sin(\frac{(2i-1)\pi}{2}) = 0,$$

and

$$n \sin \theta_{1} = n \left( \frac{\sqrt{4\lambda_{i,n} n - 1}}{2\lambda_{i,n}} \right) \quad \rightarrow \quad \frac{(2i-1)\pi}{2} \quad \text{as} \quad n \to \infty \quad (4.12)$$

so that the limit in (4.10) is \(\frac{1}{2}\). To summarize, we have shown that

$$\lim_{n \to \infty} \frac{1}{n} C_{i,n}^2 = \frac{1}{2} \quad (4.13)$$

for all \(i\) where

$$C_{i,n} = \left( \sum_{t=1}^{n-1} x_{t,i}^2 \right)^{1/2}. \quad (4.14)$$

Case I: \(n(\hat{\beta}-1)\)

Let \(\{e_t\}_{t=1}^{\infty}\) be a sequence of independent \(N(0, \sigma^2)\) random variables.

The results of the preceding section permit us to express \(n^{-2} \sum_{t=2}^{n} x_{t-1}^2\)

as

$$n^{-2} \sum_{i=1}^{n-1} \lambda_{i,n} z_{i,n}^2 \quad (4.15)$$

where

$$z_{i,n} = C_{i,n}^{-1} \sum_{t=1}^{n-1} x_{t,i} e_t \quad (4.16)$$

and \(\{z_{i,n}; i = 1, 2, \ldots, n-1\}\) are normal independent \((0, \sigma^2)\) random variables. By (4.14)

$$\lim_{n \to \infty} n^{-2} \lambda_{i,n} = \left( \frac{2}{(2i-1)\pi} \right)^2. \quad (4.17)$$
Having expressed the denominator of $n(\beta - 1)$ in the spectral form (4.15) we wish to find the covariance between the numerator and denominator of $n(\beta - 1)$. The normalization of the numerator of $n(\beta - 1)$ by $n^{-2}$ gives
\[ n^{-1} \sum_{t=2}^{n} e_t^2 = n^{-1} \sum_{t=2}^{n} e_t^2 = (2n)^{-1} \left( \sum_{t=1}^{n} e_t^2 \right) \]
\[ = (2n)^{-1} \left( \frac{\sum_{t=1}^{n} e_t^2}{n} \right). \]

Now $(\sigma^2)^{-1} \frac{\sum_{t=1}^{n} e_t^2}{n}$ has a chi-square distribution with one degree of freedom and
\[ (\sigma^2)^{-1} \sum_{t=1}^{n} e_t^2 = 1 + O_p \left( n^{-\frac{1}{2}} \right). \]

Thus
\[ (2n)^{-1} \left( \frac{\sum_{t=1}^{n} e_t^2}{n} \right) = 1 + \frac{1}{2} \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e_t^2 \right) + O_p \left( n^{-\frac{1}{2}} \right). \quad (4.18) \]

The limit distribution of $n(\beta - 1)$ is not symmetric about 0 since the median of a chi-square distribution with one degree of freedom is 0.455 (Fisher and Yates (1963) pg. 47).

We have, for fixed $i$,
\[ \text{Cov} \left( \frac{1}{\sqrt{n}} (e_1 + e_2 + \ldots + e_n), Z_{i,n} \right) = \frac{\sigma^2}{\sqrt{n} c_{i,n}} \sum_{t=1}^{n-1} x_{t,i} \]
\[ = \frac{\sigma^2}{\sqrt{n} c_{i,n}} \sum_{t=1}^{n-1} [\cos(\theta_i t) + (\lambda_{i,n} - 1) \frac{1}{2} \sin(\theta_i t)] \]
\[ = \frac{\sigma^2}{\sqrt{n} c_{i,n}} \sum_{t=1}^{n-1} \cos(\theta_i t) + O(1) \]
\[ (4.19) \]

where $Z_{i,n}$ was defined in (4.16) and $c_{i,n}$ in (4.14).

Now by Jolley (1961, pg. 78 no. 4.18),
\[
\frac{\sigma^2}{\sqrt{n} C_{i,n}} \sum_{t=1}^{n-1} \cos(\theta_i t) = \left( \frac{n \sigma^2}{C_{i,n}} \right) \beta_{i,n}
\]

where
\[
\beta_{i,n} = n^{-1} \cos\left( \frac{1}{2}(n+1)\theta_i \right) \sin\left( \frac{1}{2}n\theta_i \right) \csc\left( \frac{\theta_i}{2} \right)^n, \quad (4.21)
\]

Using (4.11) and (4.12) we have
\[
\lim_{n \to \infty} \cos\left( \frac{1}{2}(n+1)\theta_i \right) \sin\left( \frac{1}{2}n\theta_i \right) = \cos\left( \frac{(2i-1)\pi}{4} \right) \sin\left( \frac{(2i-1)\pi}{4} \right)
\]
\[
= \frac{1}{2}(-1)^{i+1}
\]

and
\[
\lim_{n \to \infty} n \sin\left( \frac{\theta_i}{2} \right) = \frac{(2i-1)\pi}{4}. \quad (4.22)
\]

Combining these results we have
\[
\lim_{n \to \infty} \beta_{i,n} = \frac{2(-1)^{i+1}}{(2i-1)\pi}
\]

and from (4.13)
\[
\lim_{n \to \infty} \left( \frac{n}{C_{i,n}} \right) = \sqrt{2}
\]

for all i. Let
\[
\gamma_i = \frac{2(-1)^{i+1}}{(2i-1)\pi}.
\]

Now for \( e' = (e_1, e_2, \ldots, e_n) \) we have expressed \( \sim \) as
\[
\sim = (Z_1, Z_2, \ldots, Z_n)' = M e
\]

where M is a nonsingular matrix. Now let
\[
T_n = \delta' e
\]

where
\[
\delta' = \left( \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right).
\]

Thus
\[ T_n = \tilde{\delta}^\prime M^{-1} \tilde{Z} \]

so the elements of \( \tilde{\delta}'M^{-1} \) are the covariances between \( T_n \) and \( Z_i \) and

\[ T_n = \sum_{i=1}^{n} \text{Cov}(T_n, Z_i)Z_i \].

To summarize, we have shown that the numerator of \( n(\hat{\delta}-1) \) is

\[ \frac{1}{2}(T_n^2 - \sigma^2) + O_p(n^{-\frac{1}{2}}) \]

where

\[ T_n = \frac{1}{\sqrt{n}}(e_1 + e_2 + \ldots + e_n) = \sum_{i=1}^{n-1} \sqrt{n}(\beta_{i, n}/C_{i, n})Z_i, n + O_p(n^{-\frac{1}{2}}) \]

\[ (Z_{i, n}', Z_{2, n}', \ldots, Z_{n-1, n}') \sim N(0, \sigma^2 I) \] for all \( n \)

and

\[ \lim_{n \to \infty} (\sqrt{n}(\beta_{i, n}/C_{i, n})) = \sqrt{2} \gamma_1 \]

for all \( i \). The denominator of \( n(\hat{\delta}-1) \) has been written as

\[ \sum_{i=1}^{n-1} n^{-2} \lambda_i, n \]

where, by (4.17), \( \lim_{n \to \infty} n^{-2} \lambda_i, n = \gamma_i^2 \) for each \( i \).

Case II: \( n(\hat{\delta}_\mu - 1) \)

We now obtain a representation for the numerator and denominator of \( n(\hat{\delta}_\mu - 1) \) similar to the representations derived for \( n(\hat{\delta}-1) \). Using the representation

\[ T_n = \frac{1}{\sqrt{n}}(e_1 + e_2 + \ldots + e_n) \]

and (4.25) we write the numerator of \( n(\hat{\delta}^\prime_\mu - 1) \) as

\[ \frac{1}{2}(T_n^2 - \sigma^2) - \frac{(n-1)}{n} e \bar{Y}(-1) + O_p(n^{-1/2}) \]

where
\[ \bar{Y}_{(-1)} = (n-1)^{-2} \left( \sum_{j=2}^{n} e_j \right) \left( \sum_{i=1}^{n-1} e_i \right). \]

Let
\[ W_n = (n-1) - \frac{1}{2} \bar{Y}_{(-1)} = (n-1) - \frac{3}{2} \sum_{i=1}^{n-1} e_i. \quad (4.29) \]

The normalized numerator of \( n(\hat{\beta}_1 - 1) \) may then be expressed as
\[
\frac{1}{2} [T_n^2 - \sigma^2] + o_p \left( n^{-\frac{1}{2}} \right) - \left( \frac{n-1}{n} \right)^2 (n-1)^{-2} (\sqrt{n} T_n - e_1) (n-1)^{\frac{3}{2}} W_n
\]
\[
= \frac{1}{2} [T_n^2 - \sigma^2] - T_n W_n + o_p \left( n^{-\frac{1}{2}} \right). \quad (4.30)\]

The normalized denominator of \( n(\hat{\beta}_1 - 1) \) is
\[
n^{-2} \left( \sum_{t=2}^{n} Y_{t-1}^2 - (n-1)^{-1} \left( \sum_{t=2}^{n} Y_{t-1}^2 \right) \right)
\]
\[
n^{-2} \sum_{t=2}^{n} Y_{t-1}^2 - (n-1)^{-1} T_n^2 \bar{Y}_{(-1)}
\]
\[
n^{-2} \sum_{i=1}^{n-1} i i_i, n^2 i_i, n - W_n^2 + o_p \left( n^{-1/2} \right). \quad (4.31)\]

We wish to express (4.30) and (4.31) in terms of the \( Z_{i,n} \) of (4.16) and use this representation in simulation. To do this, we require

\[ \text{Cov}(Z_{i,n}, W_n). \]

Now
\[
\text{Cov}(Z_{i,n}, W_n) = \text{Cov} \left( C_{i,n}^{-1} \sum_{t=1}^{n-1} t e_t, (n-1) - \frac{3}{2} [(n-1)e_1 + (n-2)e_2], \ldots + e_{n-1} \right)
\]
\[
= \frac{\sigma^2 C_{i,n}^{-1} (n-1)}{2} \text{Cov} \left( t \cos(\theta_i t) + (n \lambda_{i,n} - 1)^{-1} \frac{1}{2} \sin(\theta_i t) \right)
\]
\[
= \frac{3}{2} \sum_{t=1}^{n-1} \text{Cov} \left( n \cos(\theta_i t) - \sum_{t=1}^{n-1} t \cos(\theta_i t) \right) + o(\frac{1}{n})
\]
\[
= \frac{3}{2} \left( \frac{n \sigma^2 C_{i,n}^{-1} (n-1)}{2} \right) \text{Cov} \left( \frac{\theta_i}{2} \right) + \left( \frac{1 - \cos(n \theta_i)}{4 \sin^2(\frac{\theta_i}{2})} \right)
\]
\[+ O\left(\frac{1}{n}\right),\] (4.32)

where we have used Jolley (1961, pg. 80 no. 428). Letting

\[\alpha_{i,n} = (2n)^{-1}\sin\left(\frac{2n-1}{2}\theta_{i}\right)\csc\left(\frac{\theta_{i}}{2}\right)\]

and

\[\eta_{i,n} = \frac{1}{4}n^{-2}\csc^{2}\left(\frac{\theta_{i}}{2}\right)(1-\cos(n\theta_{i}))\]

Expression (4.32) becomes

\[\text{Cov}(Z_{i,n}, W_{n}) = \sigma^{2}c_{i,n}^{-1}(n-1) - \frac{3}{2}(n^{2})(\beta_{i,n} - \alpha_{i,n} + \eta_{i,n}) + O\left(\frac{1}{n}\right).\] (4.33)

Now

\[\lim_{n \to \infty} n\sin\left(\frac{\theta_{i}}{2}\right) = \frac{(2i-1)n}{4}\] (4.34)

and

\[\lim_{n \to \infty} \sin((n - \frac{1}{2})\theta_{i}) = \sin\left(\frac{(2i-1)n}{2}\right) = (-1)^{i+1}\] (4.35)

so that

\[\lim_{n \to \infty} \alpha_{i,n} = \frac{2}{(2i-1)n}(-1)^{i+1} = \gamma_{i}\] (4.36)

and

\[\lim_{n \to \infty} \eta_{i,n} = \frac{1}{4}(\frac{1}{(2i-1)n})^{2} = \gamma_{i}^{2} = \gamma_{i}^{2}.\] (4.37)

Since

\[\lim_{n \to \infty} n\ c_{i,n}^{-1} = \sqrt{2}\]

and

\[\lim_{n \to \infty} \beta_{i,n} = \gamma_{i}\]

we see that the limit of Expression (4.33) is

\[\lim_{n \to \infty} \text{Cov}(Z_{i,n}, W_{n}) = \sqrt{2}\sigma^{2}\gamma_{i}^{2}\].

That is, we can express the normalized numerator of \(n(\hat{\beta} - \beta)\) as

\[\frac{1}{2}\left[n^{2} - \sigma^{2}\right] - T_{\hat{\beta}}W_{n} + O\left(\frac{1}{n^{2}}\right)\] (4.38)
and the normalized denominator of \( n(\hat{\beta}_n - 1) \) as

\[
\frac{n-2}{n} \sum_{i=1}^{n-1} \lambda_i, n \gamma_i^2, n - W_n - O_p(\frac{1}{n})
\]

(4.39)

where, by arguments analogous to those for \( T_n \),

\[
W_n = \sum_{i=1}^{n-1} \frac{1}{\sqrt{n}} (\beta_{i,n} - \alpha_{i,n} + \eta_{i,n}) Z_{i,n} + O_p(\frac{1}{n})
\]

(4.40)

and

\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} (\beta_{i,n} - \alpha_{i,n} + \eta_{i,n}) = \sqrt{2} \gamma_i^2, i = 1, 2, \ldots
\]

(4.41)

Case III: \( n(\hat{\beta}_n - 1) \)

The normalized numerator of \( n(\hat{\beta}_n - 1) \) is

\[
\frac{1}{n} \sum_{t=2}^{n} e_t Y_{t-1} - \frac{1}{n} \sum_{t=2}^{n} e_t Y_{t-1} - \frac{12}{n(n-1)(n-2)} \sum_{t=2}^{n} (t-\frac{n}{2}) e_t Y_{t-1}
\]

(4.42)

Now

\[
\frac{1}{\sqrt{n}} \left( \frac{1}{n} \sum_{t=2}^{n} (t-\frac{n}{2}) e_t \right) = \frac{1}{\sqrt{n}} \left( \sqrt{n-1} W_n + \sqrt{n} \left( \frac{n}{n-1} \right) T_n - \frac{\sqrt{n}}{n-1} \right)
\]

(4.43)

where \( T_n \) is defined in (4.26) and \( W_n \) is defined in (4.29). Therefore we may rewrite (4.42) as

\[
\frac{1}{2} \left( \frac{T_n^2}{n} - \frac{\sigma^2}{n} \right) - T_n W_n - 6 \left( -W_n + \frac{1}{2} \right) T_n (V_n) + O_p(\frac{1}{n})
\]

(4.44)

where

\[
V_n = \frac{2}{n(n-2)} \sum_{t=2}^{n} (t-\frac{n}{2}) y_{t-1} = \frac{1}{n(n-2)} \sum_{j=1}^{n-1} (n-j)(j-1)e_j
\]

(4.45)

The covariance between \( V_n \) and \( Z_{i,n} \) is found by noting that
\[
\text{Cov}(c_i^t, x_t, i_t, n) = n^{-1} \sum_{t=1}^{n-1} (n-j)(j-1)e_j
\]
\[
= \sigma^2/n c_i^t, n^{-3} \sum_{t=1}^{n-1} (n-t)(t-1)(\cos(\theta_1 t) + (4\lambda_1, n-1)^{-1} \frac{1}{2} \sin(\theta_1 t)). (4.46)
\]

Now
\[
|n^{-3} \sum_{t=1}^{n-1} (n-t)(t-1)(4\lambda_1, n-1)^{-1} \frac{1}{2} \sin(\theta_1 t)|
\]
\[
\leq n^{-3} (4\lambda_1, n-1)^{-1} \frac{1}{2} n^{-1} \sum_{t=1}^{n-1} (n-t)(t-1) = O(\frac{1}{n}). \quad (4.47)
\]
Also
\[
n^{-3} \sum_{t=1}^{n-1} (n-t)(t-1)\cos(\theta_1 t) = n^{-2} \sum_{t=1}^{n-1} t \cos(\theta_1 t)
\]
\[
- n^{-3} \sum_{t=1}^{n-1} t^2 \cos(\theta_1 t) + O(\frac{1}{n}). \quad (4.48)
\]

Letting
\[
\gamma_1, n = n^{-3} \sum_{t=1}^{n-1} t^2 \cos(\theta_1 t)
\]
we see that
\[
\lim_{n \to \infty} \gamma_1, n = \lim_{n \to \infty} \sum_{t=1}^{n-1} \frac{\cos(ne_1/n)}{n} = \int_0^1 x^2 \cos(\frac{21-1}{2}n)dx
\]
\[
= \gamma_1 - 2\gamma_1^3. \quad (4.50)
\]

Using (4.32), (4.47), (4.48), and (4.49) Expression (4.46) becomes
\[
\text{Cov}(Z_i, V_n) = \sigma^2/n c_i^t, n^{-1} [\alpha_i, n - \gamma_i, n - \gamma_i, n] + O(\frac{1}{n}). \quad (4.51)
\]
Using (4.13), (4.36), (4.37), and (4.50) the limit in (4.51) is
\[
\lim_{n \to \infty} \text{Cov}(Z_i, V_n) = \sqrt{2} \sigma^2[\gamma_1 - \gamma_1^2 - \gamma_1^3 + 2\gamma_1^3] = \sigma^2[\sqrt{2} \gamma_1^3 - \sqrt{2} \gamma_1^3]. \quad (4.52)
\]
The normalized denominator of \( n(c_\tau - 1) \) is
\[
\frac{1}{n^2} \sum_{t=2}^{n} \left( \frac{1}{n^2(n-1)} \right) \left( \Sigma_{t=2}^{n} Y_{t-1} \right)^2 - \frac{12}{n^3(n-1)(n-2)} \left( \Sigma_{t=2}^{n} (t-1-\frac{n}{2}) Y_{t-1} \right)^2 = \\
\frac{1}{n^2} \sum_{i=1}^{n-1} \Sigma_{i,1,n}^2 Z_i, n - W_i, n - 3V_i, n + O_p(n^{-\frac{1}{2}}).
\]

(4.53)

To summarize, the numerator of \(n(\hat{\beta}_r - 1)\) is \(N_r\), say, where

\[
N_r = \left( \frac{1}{2} T_n - W_n \right) \left( T_n - 6V_n \right) - \frac{\sigma^2}{2} + O_p(n^{-\frac{1}{2}}) 
\]

(4.54)

and the denominator is \(D_r\), say, where

\[
D_r = n^{-2} \sum_{i=1}^{n-1} \Sigma_{i,1,n}^2 Z_i, n - W_i, n - 3V_i, n + O_p(n^{-\frac{1}{2}})
\]

where \(T_n\) is defined in (4.26), \(W_n\) is defined in (4.29) and, by arguments analogous to those for \(T_n\),

\[
V_n = \sum_{i=1}^{n-1} \Sigma_{i,1,n}^2 (\alpha_i, n - \eta_i, n - \gamma_i, n) Z_i, n + O_p(n^{-\frac{1}{2}}). 
\]

(4.55)

Convergence in Distribution

In the previous three sections we have shown that \(n(\hat{\beta} - 1), n(\hat{\beta}_\mu - 1),\) and \(n(\hat{\beta}_r - 1)\) are expressible up to \(O_p(n^{-\frac{1}{2}})\) as functions of the random variables \(n^{-2} \sum_{i=1}^{n-1} \Sigma_{i,1,n}^2 Z_i, n, T_n, W_n,\) and \(V_n\). Our goal in this section will be to define limit laws for our estimators.

Let \([Z_i]_{i=1}^\infty\) be a sequence of independent normal \((0, \sigma^2)\) random variables. Define

\[
T_n^* = \sum_{i=1}^{n-1} \sqrt{n} (\beta_i, n / C_i, n) Z_i, n 
\]

(4.56)

\[
W_n^* = \sum_{i=1}^{n-1} \sqrt{n} C_i, n (\beta_i, n - \alpha_i, n + \eta_i, n) Z_i, n 
\]

(4.57)

\[
V_n^* = \sum_{i=1}^{n-1} \sqrt{n} C_i, n (\alpha_i, n - \gamma_i, n) Z_i, n 
\]

(4.58)

and

\[
\Gamma_n^* = n^{-2} \sum_{i=1}^{n-1} \Sigma_{i,1,n}^2 Z_i, n 
\]

(4.59)
By comparing these to (4.26), (4.40), (4.55) and (4.15) we see that, except for terms of $O_p(n^{-1/2})$, the quantities (4.56) through (4.59) have the same distribution as $T_n$, $W_n$, $V_n$, and $n^{-2} \sum_{i=1}^{n-1} \lambda_i Z_i^2$, respectively. Define

$$\eta' = (T, W, V, \Gamma) \quad (4.60)$$

where

$$T = \sum_{i=1}^{\infty} \gamma_1 Z_i \quad (4.61)$$

$$W = \sum_{i=1}^{\infty} \gamma_i^2 Z_i \quad (4.62)$$

$$V = \sum_{i=1}^{\infty} (\gamma_1 Z_i^3 - 2 \gamma_i^2 Z_i) \quad (4.63)$$

and

$$\Gamma = \sum_{i=1}^{\infty} \gamma_i^2 Z_i^2 \quad (4.64)$$

where $\gamma_1 = \frac{2(-1)^{1+1}}{(2i-1)n}$.

**Lemma 4.1.** The vector $\eta$ in (4.60) is a well-defined vector random variable.

**Proof:** Since $\sum_{i=1}^{\infty} \gamma_1^k < \infty$ for $k \geq 1$ we see, for example, that

$$\lim_{n \to \infty} \sum_{i=1}^{n} \text{Var}(\gamma_{i}^2 Z_i^2) = \lim_{n \to \infty} \sum_{i=1}^{n} \gamma_i^4 < \infty$$

so the variance of $\Gamma$ exists and is finite. Variances exist for all other elements of $\eta$. Further,

$$E(\eta') = (0, 0, 0, \frac{1}{2}).$$
By Tucker (1967) Theorem 2 page 110 the elements of \( \mathbb{J} \) are infinite sums which converge almost surely. Thus \( \mathbb{J} \) has a distribution function and is a well-defined vector random variable.

We now wish to show that the random vector

\[
\mathbb{J}_n' = (\mathbb{J}_n^*, \mathbb{J}_n^*, \mathbb{J}_n^*, \mathbb{J}_n^*)
\]

converges in law to \( \mathbb{J} \). We first prove a useful lemma.

**Lemma 4.2.** Let \( \{Z_i\}_{i=1}^\infty \) be a sequence of independent \((0, \sigma^2)\) random variables. Let \( \{w_i; i = 1, 2, \ldots\} \) be a sequence of real numbers and let \( \{w_{i,n}; i = 1, 2, \ldots, n-1; n = 1, 2, \ldots\} \) be a triangular array of real numbers.

If

\[
\sum_{i=1}^\infty w_i^2 < \infty,
\]

\[
\lim_{n \to \infty} \sum_{i=1}^n w_{i,n}^2 = \sum_{i=1}^\infty w_i^2,
\]

and

\[
\lim_{n \to \infty} w_{i,n} = w_i,
\]

then \( \sum_{i=1}^\infty w_i Z_i \) is a well-defined random variable and

\[
\sum_{i=1}^{\infty} w_{i,n} Z_i \overset{p}{\rightarrow} \sum_{i=1}^{\infty} w_i Z_i.
\]

**Proof:** By arguments analogous to those in Lemma 4.1, \( \sum_{i=1}^\infty w_i Z_i \) is a well-
defined random variable. Let \( s > 0 \) be given. Since \( \sum_{i=1}^{\infty} w_i^2 < \infty \) and
\[
\lim_{n \to \infty} \sum_{i=1}^{n} w_{i,n}^2 = \sum_{i=1}^{\infty} w_i^2
\]
we can pick an \( M \) such that
\[
\sigma^2 \sum_{i=M}^{\infty} w_i^2 < \varepsilon/10 \quad (4.68)
\]
and
\[
\sigma^2 \left| \sum_{i=1}^{n} w_{i,n}^2 - \sum_{i=1}^{\infty} w_i^2 \right| < \varepsilon/10 \quad (4.69)
\]
for all \( n > M \). Further, since \( \lim_{n \to \infty} w_{i,n} = w_i \) we can find an \( N_0 > M \) such that \( n > N_0 \) implies
\[
\sigma^2 \sum_{i=1}^{M} (w_{i,n} - w_i)^2 < \varepsilon/10. \quad (4.70)
\]
and
\[
\sigma^2 \sum_{i=1}^{M} (w_{i,n}^2 - w_i^2) < \varepsilon/10. \quad (4.71)
\]
For \( n > N_0 \),
\[
\text{Var}(\sum_{i=1}^{n} w_{i,n} Z_i - \sum_{i=1}^{\infty} w_i Z_i) =
\]
\[
\sigma^2 \sum_{i=1}^{M} (w_{i,n} - w_i)^2 + \sigma^2 \sum_{i=M+1}^{n} w_{i,n}^2 
\]
\[
\leq \varepsilon/10 + \sigma^2 \left[ \sum_{i=1}^{n} w_{i,n}^2 - \sum_{i=1}^{\infty} w_i^2 \right] + \sum_{i=1}^{M} w_i^2 - \sum_{i=1}^{M} w_{i,\infty}^2 + \sum_{i=M+1}^{\infty} w_i^2 
\]
\[
\leq \varepsilon/10 + \varepsilon/10 + \varepsilon/10 + \varepsilon/10 = \frac{2\varepsilon}{5}
\]
by (4.68), (4.69), (4.70), and (4.71). Now
\[
\text{Var}(\sum_{i=M+1}^{\infty} w_{i,n} Z_i) < \varepsilon/10
\]
so
\[
\Var\left( \sum_{i=1}^{n} w_i Z_i - \sum_{i=1}^{M} w_i Z_i \right) \leq 2 \Var\left( \sum_{i=1}^{n} w_i Z_i - \sum_{i=1}^{M} w_i Z_i \right) \\
+ 2 \Var\left( \sum_{i=M+1}^{\infty} w_i Z_i \right) < \frac{\epsilon \sigma^2}{5} + \frac{\epsilon}{5} = \epsilon.
\]

By Chebyshev's inequality
\[
\sum_{i=1}^{n} w_i Z_i \xrightarrow{P} \sum_{i=1}^{\infty} w_i Z_i.
\]

To apply this lemma to \( T_n \) we note that, for example,
\[
\Var(T_n) = \sigma^2 \sum_{i=1}^{n-1} \frac{(\beta_i n)^2}{\xi_i n}
\]
but
\[
\lim_{n \to \infty} \Var(T_n) = \lim_{n \to \infty} \Var(T_n) = \sigma^2
\]
since \( T_n = \frac{1}{n} (e_1 + e_2 + \ldots + e_n) \). Thus we let
\[
\frac{w_i}{n} = \frac{\beta_i n}{\xi_i n}
\]
and from (4.27)
\[
w_i = \lim_{n \to \infty} \frac{w_i}{n} = \sqrt{2} \gamma_i = \sqrt{2} \frac{2(-1)^{i+1}}{(2i-1)n}
\]
which is condition (4.67). Using Jolley (1961, pg. 56) and (4.61) we find
\[
\Var(T) = 2 \sum_{i=1}^{\infty} \gamma_i \sigma^2 = \sigma^2 \lim_{n \to \infty} \sum_{i=1}^{n-1} \frac{\beta_i n^2}{\xi_i n}
\]
Thus (4.66) holds and \( T_n \xrightarrow{P} T \). The analogous convergence results for \( W_n^*, V_n^*, \text{ and } \Gamma_n^* \) follow by first noting that
\[
\lim_{n \to \infty} \Var(W_n) = \lim_{n \to \infty} (n-1)^{-3} \sum_{i=1}^{n-1} (n-1)^2 \sigma^2 = \frac{\sigma^2}{3},
\]
\[
\lim \text{Var}(V_n) = \lim_{n \to \infty} n^{-5} \sum_{i=1}^{n-1} (n-1)^2 \sigma^2 = \\
= \lim_{n \to \infty} n^{-5} \sum_{i=1}^{n} (n^2 i^2 + \frac{1}{6} - 2ni^3 \sigma^2 = \frac{\sigma^2}{30},
\]
and
\[
\lim \text{Var}(\Gamma_n) = \lim_{n \to \infty} n^{-4} (\frac{\sigma^2}{3}) n(n-1)(n^2-n+1) = \frac{\sigma^4}{3},
\]
where we have used Expressions (4.29) and (4.45) and Table 3.4. By Jolley (1961, pg. 5), we find
\[
\sum_{i=1}^{\infty} \gamma_i^2 = \frac{1}{2}, \quad \sum_{i=1}^{\infty} \gamma_i^3 = \frac{1}{4}, \quad \sum_{i=1}^{\infty} \gamma_i^4 = \frac{1}{6}, \quad \sum_{i=1}^{\infty} \gamma_i^5 = \frac{5}{48}, \quad \text{and} \quad \sum_{i=1}^{\infty} \gamma_i^6 = \frac{1}{15}
\]
so that
\[
2 \sum_{i=1}^{\infty} \gamma_i^4 \sigma^2 = \frac{\sigma^2}{3},
\]

\[
\sum_{i=1}^{\infty} ((\bar{\gamma}_i)^3 - \sqrt{2} \gamma_i^2 \gamma_i^2 \sigma^2 = \frac{\sigma^2}{30},
\]
and
\[
\sum_{i=1}^{\infty} 2\gamma_i^4 \sigma^4 = \frac{\sigma^4}{3}
\]
which, using (4.62), (4.63), and (4.64) are the summations of interest for W, V, and Γ. Thus Conditions (4.66) and (4.67) are satisfied and by Lemma 4.2 we have proved the following theorem.

**Theorem 4.1.** For \( \Psi_n \) and \( \Psi \) as defined in (4.65) and (4.60) respectively, \( \Psi_n \) converges in probability to \( \Psi \).

Recall that \( (T_n, W_n, V_n, \Gamma_n) \) and \( (T_n, W_n, V_n, \Gamma_n) \) have the same limit distribution. Our estimators \( n(\hat{\beta}-1), n(\hat{\rho}_\pi-1), \) and \( n(\hat{\rho}_\pi-1) \) are ratios of quadratic forms. The numerator quadratic forms are given by
Expressions (4.30), (4.38), and (4.54). The denominator quadratic forms are given in expressions (4.15), (4.13), and (4.53) and are positive definite.

Corollary 4.1.

\[ n(\delta - 1) \xrightarrow{\mathcal{L}} \frac{\frac{1}{2}(T^2 - \sigma^2)}{\Gamma} \]

\[ n(\delta^*_\mu - 1) \xrightarrow{\mathcal{L}} \frac{\frac{1}{2}(T^2 - \sigma^2) - TW}{\Gamma - W^2} \]

and

\[ n(\delta^*_\tau - 1) \xrightarrow{\mathcal{L}} \frac{\frac{1}{2} (T - W)(T - 6V) - \frac{1}{2} \sigma^2}{\Gamma - W^2 - 3V^2} \]

where \( \{Z_i\}_{i=1}^\infty \) is a sequence of normal independent \( (0, \sigma^2) \) random variables,

\[ \Gamma = \sum_{i=1}^\infty \gamma_{1i}^2 \]

\[ T = \sqrt{2} \sum_{i=1}^\infty \gamma_{1i} Z_i \]

\[ W = \sum_{i=1}^\infty \sqrt{2} \gamma_{1i}^2 Z_i \]

\[ V = \sum_{i=1}^\infty [(\sqrt{2} \gamma_{1i})^3 - \sqrt{2} \gamma_{1i}^2 Z_i] \]

and

\[ \gamma_{1i} = \frac{2}{(2i-1)!} (-1)^{i+1} \]

Proof:

The denominator quadratic forms are continuous functions of \( \gamma \) which have probability 1 of being positive. Since \( \gamma_n \) converges in law to \( \gamma \) and since the ratios of the limit quadratic forms are continuous functions on a set which contains \( \gamma \) with probability 1, the estimators con-
verge as stated.

Corollary 4.2.

\[
\tau \xrightarrow{\mathcal{L}} \frac{1}{2}(T^2 - \sigma^2) \sqrt{\frac{1}{T}}
\]

\[
\tau_{\mu} \xrightarrow{\mathcal{L}} \frac{1}{2}(T^2 - \sigma^2) - TW \sqrt{\frac{1}{T} - \frac{1}{W^2}}
\]

and

\[
\tau_{\tau} \xrightarrow{\mathcal{L}} \frac{\left(\frac{1}{2}T - W\right)(T - 6V) - \frac{1}{2}c^2}{\sqrt{\frac{1}{T} - \frac{1}{W^2} - \frac{1}{3V^2}}}
\]

where \(\{Z_i\}_{i=1}^\infty\) is a sequence of normal independent \((0, \sigma^2)\) random variables,

\[
\Gamma = \sum_{i=1}^{\infty} \gamma_i Z_i^2
\]

\[
T = \sum_{i=1}^{\infty} \sqrt{2} \gamma_i Z_i
\]

\[
W = \sum_{i=1}^{\infty} \sqrt{2} \gamma_i Z_i
\]

\[
V = \sum_{i=1}^{\infty} \left(\sqrt{2} \gamma_i\right)^3 - \sqrt{2} \gamma_i^2 Z_i
\]

and

\[
\gamma_i = \frac{2}{(2i-1)\pi} (-1)^{i+1}
\]

Proof:

We show the proof only for \(\tau\) because the proofs for \(\tau_{\mu}\) and \(\tau_{\tau}\) are analogous to that for \(\tau\).

We note that the normalized denominator for \(n(\hat{\sigma} - 1)\) is

\[
\frac{1}{n^2\sigma^2} \left(\sum_{t=2}^{n} \gamma_t^2\right)
\]

and that from (2.5),
\[
\tau = \frac{n(\hat{\beta}-1) \sum_{t=2}^{n} \frac{Y_t^2}{(n\sigma^2)}}{\sqrt{\frac{1}{n} \sum_{t=2}^{n} (Y_t - \hat{\beta}Y_{t-1})^2}} \sqrt{\frac{1}{n} \sum_{t=2}^{n} \left( \frac{Y_t^2}{(n-1)\sigma^2} \right)} \quad (4.74)
\]

Since
\[
(n-1)^{-1} \sum_{t=2}^{n} (Y_t - \hat{\beta}Y_{t-1})^2 = (n-1)^{-1} \left[ \sum_{t=2}^{n} e_t^2 + 2(\hat{\beta}-\rho) \sum_{t=2}^{n} e_t Y_{t-1} \right] + (\rho-\hat{\beta})^2 n \sum_{t=2}^{n} Y_{t-1}^2
\]

we see that multiplication of \(n(\hat{\beta}-1)\) by the square root of its normalized denominator yields a statistic with the same limit distribution as \(\tau\). Since this is a function of \(T_n\) and \(n \sum_{i=1}^{n} \xi_i^2, n\lambda_i^2, n\), the substitution of \(T\) and \(\Gamma\) in the formula for \(\tau\) will give the appropriate limit distribution.

\[\square\]

**Simulation**

Note that in (4.61) through (4.64) \(T, V, W\) and \(\Gamma\) were defined as infinite weighted sums of \(N(0, \sigma^2)\) random variables. For simulation purposes we must replace the infinite sums by finite sums. We shall choose the finite sums so that the first two moments of the finite sums agree with those of the infinite sums.

Using (4.61) through (4.64) the variance-covariance matrix of \((T, W, V)\) is
\[
\text{Cov}(T, W, V) = \begin{pmatrix}
2 \sum_{i=1}^\infty \gamma_i^2 & 2 \sum_{i=1}^\infty \gamma_i^3 & \sum_{i=1}^\infty (4\gamma_i^4 - 2\gamma_i^3) \\
2 \sum_{i=1}^\infty \gamma_i^3 & 2 \sum_{i=1}^\infty \gamma_i^4 & \sum_{i=1}^\infty (4\gamma_i^5 - 2\gamma_i^4) \\
\sum_{i=1}^\infty (4\gamma_i^4 - 2\gamma_i^3) & \sum_{i=1}^\infty (4\gamma_i^5 - 2\gamma_i^4) & \sum_{i=1}^\infty ((\sqrt{2}\gamma_i)^3 - \sqrt{2}\gamma_i^5) \end{pmatrix} \sigma^2(4.75)
\]

Jolley (1961, pg. 56) gives the following sums: \(\sum_{i=1}^\infty \gamma_i = \frac{1}{2}, \sum_{i=1}^\infty \gamma_i^2 = \frac{1}{2},\)

\(\sum_{i=1}^\infty \gamma_i^3 = \frac{1}{4}, \sum_{i=1}^\infty \gamma_i^4 = \frac{1}{8}, \sum_{i=1}^\infty \gamma_i^5 = \frac{5}{48}, \sum_{i=1}^\infty \gamma_i^6 = \frac{1}{15}, \sum_{i=1}^\infty \gamma_i^7 = \frac{61}{1440}\) and

\(\sum_{i=1}^\infty \gamma_i^8 = \frac{17}{830}.\) Substituting these results in (4.75) we obtain

\[
\text{Cov}(T, W, V) = \begin{pmatrix}
1 & 1/2 & 1/6 \\
1/2 & 1/3 & 1/12 \\
1/6 & 1/12 & 1/30
\end{pmatrix} \sigma^2. \quad (4.76)
\]

As a check of our calculations we calculate the first two moments of the numerators and denominators of the limit random variables for \(n(\hat{\rho} - 1), n(\hat{\rho} - 1),\) and \(n(\hat{\rho} - 1)\) using the above matrix and sums of powers of \(\gamma_i.\)

\[
\mathbb{E}\left[\sum_{i=1}^\infty \gamma_i^2 \right] = \sigma^2, \quad \sum_{i=1}^\infty \gamma_i^2 = \frac{\sigma^2}{2},
\]

\[
\text{Var}\left[\sum_{i=1}^\infty \gamma_i^2 \right] = 2\sigma^4, \quad \sum_{i=1}^\infty \gamma_i^4 = \frac{\sigma^4}{3},
\]

\[
\mathbb{E}\left[\frac{1}{2} \left( T^2 - \sigma^2 \right) \right] = 0,
\]

\[
\text{Var}\left[\frac{1}{2} \left( T^2 - \sigma^2 \right) \right] = \frac{1}{4} \left( 2\mathbb{E}(T^2) \right)^2 = \frac{\sigma^4}{2},
\]
\[
\text{Cov}\{\frac{1}{2} (T^2 - \sigma^2), \sum_{i=1}^{\infty} \gamma_i^2 \sigma_i^2 \} = 2 \sum_{i=1}^{\infty} \gamma_i^4 \sigma_i^4 = \frac{\sigma_i^4}{3},
\]

\[
E\{ \sum_{i=1}^{\infty} \gamma_i^2 \sigma_i^2 - W^2 \} = \frac{\sigma_i^2}{6},
\]

\[
\text{Var}\{ \sum_{i=1}^{\infty} \gamma_i^2 \sigma_i^2 - W^2 \} = \left[ 2 \sum_{i=1}^{\infty} \gamma_i^4 - 8 \sum_{i=1}^{\infty} \gamma_i^6 + 2(\frac{1}{3}) \sigma_i^4 \right] = \frac{\sigma_i^4}{45},
\]

\[
E\{ \frac{1}{2} (T^2 - \sigma^2) - TW \} = -\frac{\sigma_i^2}{2},
\]

\[
\text{Var}\{ \frac{1}{2} (T^2 - \sigma^2) - TW \} = E\{ (\frac{1}{4} T^4 - T^3 W + T^2 W^2) \} = \frac{\sigma_i^4}{12},
\]

\[
\text{Cov}\{ \sum_{i=1}^{\infty} \gamma_i^2 Z_i^2 - W^2, \frac{1}{2} (T^2 - \sigma^2) - TW \} = 2 \sum_{i=1}^{\infty} \gamma_i^4 - 4 \sum_{i=1}^{\infty} \gamma_i^5 - \frac{\gamma_i^4}{4} + \frac{1}{3} = 0,
\]

\[
E\{ \sum_{i=1}^{\infty} \gamma_i^2 Z_i^2 - W^2 - 3V^2 \} = \frac{\sigma_i^2}{15},
\]

\[
E\{ \sum_{i=1}^{\infty} \gamma_i^2 Z_i^2 - W^2 - 3V^2 \} = \frac{11 \sigma_i^4}{6300},
\]

\[
E\{ (\frac{1}{2} T - W)(T - 6V) - \frac{\sigma_i^2}{2} \} = -\frac{\sigma_i^2}{2},
\]

\[
\text{Var}\{ (\frac{1}{2} T - W)(T - 6V) - \frac{\sigma_i^2}{2} \} = E\{ (\frac{1}{2} T - W)(T - 6V) \} = \frac{\sigma_i^4}{60},
\]

\[
\text{Cov}\{ \sum_{i=1}^{\infty} \gamma_i^2 Z_i^2 - W^2 - 3V^2, (\frac{1}{2} T - W)(T - 6V) - \frac{\sigma_i^2}{2} \} = 0.
\]

These moments agree with those of Table 3.4.

The moments for the limit distributions of \( n(\hat{\alpha} - 1) \) and \( \tau \) depend on
\[ \sum_{i=1}^{\infty} \gamma_i^2 \text{ and } \sum_{i=1}^{\infty} \gamma_i^4. \] Thus it was decided to create a finite set of weights \( \tilde{\gamma}_i \), \( i = 1, 2, \ldots, K \) by setting the first few \( \tilde{\gamma}_i \) equal to \( \gamma_i \) and then choosing additional \( \tilde{\gamma}_i \) such that \( \tilde{\gamma}_i > 0 \),

\[ \sum_{i=1}^{\infty} \gamma_i^2 = \sum_{i=1}^{K} \tilde{\gamma}_i^2, \]

and

\[ \sum_{i=1}^{\infty} \gamma_i^4 = \sum_{i=1}^{K} \tilde{\gamma}_i^4, \]

where \( K \) is the total number of \( \tilde{\gamma}_i \)'s. A computer program was written to set \( \tilde{\gamma}_i = \gamma_i \) for \( i = 1, 2, \ldots, 7 \) and the additional \( \tilde{\gamma}_i \) were chosen to satisfy \( \tilde{\gamma}_i = a + b(i-7) \) for \( i = 8, 9, \ldots, K \) where \( a, b, \) and \( K \) were calculated by the program so that

\[ \sum_{i=1}^{K} \tilde{\gamma}_i^2 = \sum_{i=1}^{\infty} \gamma_i^2, \quad (4.77) \]

\[ \sum_{i=1}^{K} \tilde{\gamma}_i^4 = \sum_{i=1}^{\infty} \gamma_i^4, \quad (4.78) \]

and

\[ a + b(K-7) = 0. \]

The \( K \) value used was 36. Restrictions (4.77) and (4.78) insure that the simulated numerator and denominator random variables will have the same first two moments as the limit random variables. The values \( \tilde{\gamma}_i \) satisfied Equalities (4.77) and (4.78) to within \( 5 \times 10^{-13} \).

The first two moments of \( n(\hat{\rho}_\mu - 1), \tau_\mu, n(\hat{\rho}_\tau - 1), \) and \( \tau_\tau \) involve

\[ \sum_{i=1}^{\infty} \gamma_i^k \text{ for } k = 2, 3, 4, \ldots, 8. \] The weights \( \tilde{\gamma}_i \) were judged unsatisfactory
for these cases because \( \sum_{i=1}^{K} \gamma_i^3 \) was not sufficiently close to \( \sum_{i=1}^{\infty} \gamma_i^3 \). A new set of weights \( \tilde{\gamma}_1 \) was created by letting \( \tilde{\gamma}_1 = \gamma_1 \) for \( i = 1, 2, 3, \ldots, 14 \), choosing a value \( \tilde{\gamma}_{15} \), and using a modification of the previously mentioned computer program to generate weights \( \tilde{\gamma}_i \), \( i = 16, 17, 18, \ldots, K' \) such that \( \sum_{i=1}^{K'} \tilde{\gamma}_i = \sum_{i=1}^{\infty} \gamma_i^3 \), and \( \sum_{i=1}^{K'} \gamma_i = \sum_{i=1}^{\infty} \gamma_i \). By letting \( \tilde{\gamma}_i \) range over a grid of values, a set of \( \tilde{\gamma}_i \) was found for which

\[
\left| \sum_{i=1}^{K'} \tilde{\gamma}_i^k - \sum_{i=1}^{\infty} \gamma_i^k \right| < 5 \times 10^{-10}
\]

for \( k = 2, 3, \ldots, 8 \). The value of \( K' \) was 72.

The possibility of redoing the simulation of the limit distribution for \( n(\beta - 1) \) using these new weights \( \tilde{\gamma}_i \) was considered. Since the moments of interest in this case depend only on \( \sum_{i=1}^{\infty} \gamma_i^2 \) and \( \sum_{i=1}^{\infty} \gamma_i^4 \), and since the \( \tilde{\gamma}_i \) weights preserved these sums so accurately it was decided not to go to the extra expense of redoing that simulation.

The limit random variables of interest are functions of \( T, W, V, \) and \( \Gamma \). In the simulation, \( T, V, W, \) and \( \Gamma \) were approximated by substituting \( \tilde{\gamma}_i \) or \( \tilde{\gamma}_i \) for \( \gamma_1 \) into the formulas of Corollaries 4.1 and 4.2 with \( Z_i \)'s generated by a random number generator as described in the next chapter.
CHAPTER V. PERCENTILES OF THE DISTRIBUTIONS

In this chapter we present tables of the percentiles of the distributions of \( \hat{\rho}_1 \), \( \hat{\rho}_2 \), \( \hat{\tau}_1 \), \( \tau_\mu \), and \( \tau_\tau \). In Table 5.1 we list the number of samples \( m \) generated for each length \( n \) and the number of replications \( r \) at each \( (m,n) \) combination. An exception to this table is the limit \( (n = \infty) \) distribution of \( \hat{\rho}_\tau \) and \( \tau_\tau \) for which \( m = 2^{40,000} \). Thus a replication is the creation of \( m \) time series of length \( n \) and the corresponding estimators.

Table 5.1 Parameters of Monte Carlo study.

<table>
<thead>
<tr>
<th>n</th>
<th>25</th>
<th>25</th>
<th>50</th>
<th>50</th>
<th>100</th>
<th>100</th>
<th>250</th>
<th>500</th>
<th>750</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>50000</td>
<td>10000</td>
<td>50000</td>
<td>10000</td>
<td>50000</td>
<td>10000</td>
<td>20000</td>
<td>10000</td>
<td>10000</td>
<td>100000</td>
</tr>
<tr>
<td>r</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

For cases with \( 25 \leq n \leq 750 \) a sequence of \( n \) independent normal \((0,1)\) variables was generated using the generator Super Duper from McGill University. Calling the normal independent variates \( e_t \), the time series \( Y_t \) was formed as

\[
Y_1 = e_1
\]

\[
Y_t = Y_{t-1} + e_t, \quad t = 2,3,...,n.
\]

The estimators \( n(\hat{\rho}_1), n(\hat{\rho}_2), n(\hat{\tau}_1), \tau_\mu, \) and \( \tau_\tau \) were calculated for the series \( \{Y_t\} \) and recorded on magnetic tape. Note that for a series of length \( n \), the estimators are given by regressions involving columns of length \( n-1 \). The procedure was then replicated \( m \) times. Next the tape was sorted using the FORTRAN subroutine FSORT with a modified
output subroutine to print the percentiles.

Once sufficiently accurate estimates of the percentiles were available, the magnetic tape phase of the study was replaced. The new procedure involved creating a finely divided histogram which covered the range of the percentiles. The histograms contained 800 cells. Percentiles were obtained by interpolating from the histogram.

For the limit case, the program first generated the appropriate \( \gamma \) or \( \overline{\gamma} \) weights, calculating the parameters of the linear extension internally. An appropriate number of normal (0,1) variables \( Z_i \) was then generated using Super Duper. The limit distributions of the estimators were then simulated using the statistics given in Chapter 4 (substituting \( \gamma \) or \( \overline{\gamma} \) for \( \gamma \) in the appropriate formulas). The histogram method of percentile determination was used from the outset for the limit case.

For each of the six estimators \( n(\hat{\lambda} - 1) \), \( n(\hat{\lambda} - 1) \), \( n(\hat{\tau} - 1) \), \( \tau \), \( \tau \), and \( \tau \), at sample sizes of 25, 50, 100, 250, 500, 750, and \( \infty \) (the limit case) estimates of the eight percentiles .01, .025, .05, .10, .90, .95, .975, and .99 of the distributions were constructed. For each estimator each percentile \( p_i \), \( i = 1, 2, 3, \ldots, 8 \) was plotted against \( n \). In most cases it appeared that a function of the form

\[
p_i = a_i + \frac{b_i}{n}
\]

(5.1)

where \( a_i \) and \( b_i \) are real constants would approximate the observed percentiles well. For the .01, .025, .05, and .10 percentiles of \( n(\hat{\tau} - 1) \) and \( \tau \), equations of the form

\[
p_i = a_i + \frac{b_i}{\sqrt{n}} + \frac{c_i}{n}
\]

(5.2)
seemed to provide a better fit and were used in place of (5.1).

The following tables\(^1\) were constructed by fitting regressions to the observed percentiles from the Monte Carlo study. The .01, .025, .05, and .10 values for \(n(\hat{\beta}_T-1)\) and \(\tau_T\) are the predicted values from regression equations of the form given in (5.2). The other values are the predicted values from regression equations of the form given in (5.1). The regression equations were estimated using generalized least squares where the weights used in fitting the regression equations were the numbers of samples used to determine the estimated percentiles. The sample size \(n\) is listed on the left of each table while the regression estimates of the percentiles appear in the body of the table. The numbers .01, .025, .05, and .10 listed across the top of the table are the probabilities of obtaining an estimate less than the value in the table, given that \(\rho = 1\). Note that Table 5.2 gives empirical percentiles for the estimators normalized by the series length \(n\).

\(^1\) The tables given in Fuller (1976) differ slightly from those presented here because the tables in Fuller were constructed from a smaller number of observations.
Table 5.2. Empirical percentiles for estimators of $\rho$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n(\hat{\rho}-1)$</th>
<th>$n(\hat{\rho}_h-1)$</th>
<th>$n(\hat{\rho}_T-1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>-11.87</td>
<td>-9.35</td>
<td>-7.32</td>
</tr>
<tr>
<td>50</td>
<td>-12.82</td>
<td>-9.91</td>
<td>-7.69</td>
</tr>
<tr>
<td>100</td>
<td>-13.30</td>
<td>-10.19</td>
<td>-7.88</td>
</tr>
<tr>
<td>250</td>
<td>-13.59</td>
<td>-10.36</td>
<td>-7.99</td>
</tr>
<tr>
<td>500</td>
<td>-13.69</td>
<td>-10.42</td>
<td>-8.03</td>
</tr>
<tr>
<td>750</td>
<td>-13.72</td>
<td>-10.44</td>
<td>-8.04</td>
</tr>
<tr>
<td>$\infty$</td>
<td>-13.78</td>
<td>-10.48</td>
<td>-8.07</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n(\hat{\rho}_h-1)$</th>
<th>$n(\hat{\rho}_T-1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>-17.22</td>
<td>-14.57</td>
</tr>
<tr>
<td>50</td>
<td>-18.94</td>
<td>-15.76</td>
</tr>
<tr>
<td>100</td>
<td>-19.81</td>
<td>-16.35</td>
</tr>
<tr>
<td>250</td>
<td>-20.32</td>
<td>-16.71</td>
</tr>
<tr>
<td>500</td>
<td>-20.50</td>
<td>-16.82</td>
</tr>
<tr>
<td>750</td>
<td>-20.55</td>
<td>-16.86</td>
</tr>
<tr>
<td>$\infty$</td>
<td>-20.67</td>
<td>-16.94</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n(\hat{\rho}_T-1)$</th>
<th>$n(\hat{\rho}_T-1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>-22.51</td>
<td>-19.96</td>
</tr>
<tr>
<td>50</td>
<td>-25.65</td>
<td>-22.33</td>
</tr>
<tr>
<td>100</td>
<td>-27.33</td>
<td>-23.58</td>
</tr>
<tr>
<td>250</td>
<td>-28.42</td>
<td>-24.39</td>
</tr>
<tr>
<td>500</td>
<td>-28.84</td>
<td>-24.69</td>
</tr>
<tr>
<td>750</td>
<td>-29.00</td>
<td>-24.81</td>
</tr>
<tr>
<td>$\infty$</td>
<td>-29.47</td>
<td>-25.13</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n(\hat{\rho}_T-1)$</th>
<th>$n(\hat{\rho}_T-1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>-22.51</td>
<td>-19.96</td>
</tr>
<tr>
<td>50</td>
<td>-25.65</td>
<td>-22.33</td>
</tr>
<tr>
<td>100</td>
<td>-27.33</td>
<td>-23.58</td>
</tr>
<tr>
<td>250</td>
<td>-28.42</td>
<td>-24.39</td>
</tr>
<tr>
<td>500</td>
<td>-28.84</td>
<td>-24.69</td>
</tr>
<tr>
<td>750</td>
<td>-29.00</td>
<td>-24.81</td>
</tr>
<tr>
<td>$\infty$</td>
<td>-29.47</td>
<td>-25.13</td>
</tr>
</tbody>
</table>
Table 5.3. Empirical percentiles for τ statistics.

<table>
<thead>
<tr>
<th>n</th>
<th>.01</th>
<th>.025</th>
<th>.05</th>
<th>.10</th>
<th>.90</th>
<th>.95</th>
<th>.975</th>
<th>.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>-2.66</td>
<td>-2.26</td>
<td>-1.95</td>
<td>-1.60</td>
<td>.92</td>
<td>1.33</td>
<td>1.70</td>
<td>2.16</td>
</tr>
<tr>
<td>50</td>
<td>-2.62</td>
<td>-2.25</td>
<td>-1.95</td>
<td>-1.61</td>
<td>.91</td>
<td>1.31</td>
<td>1.66</td>
<td>2.07</td>
</tr>
<tr>
<td>100</td>
<td>-2.60</td>
<td>-2.24</td>
<td>-1.95</td>
<td>-1.61</td>
<td>.90</td>
<td>1.29</td>
<td>1.64</td>
<td>2.03</td>
</tr>
<tr>
<td>250</td>
<td>-2.58</td>
<td>-2.24</td>
<td>-1.95</td>
<td>-1.62</td>
<td>.89</td>
<td>1.28</td>
<td>1.63</td>
<td>2.01</td>
</tr>
<tr>
<td>500</td>
<td>-2.58</td>
<td>-2.24</td>
<td>-1.95</td>
<td>-1.62</td>
<td>.89</td>
<td>1.28</td>
<td>1.62</td>
<td>2.00</td>
</tr>
<tr>
<td>750</td>
<td>-2.58</td>
<td>-2.24</td>
<td>-1.95</td>
<td>-1.62</td>
<td>.89</td>
<td>1.28</td>
<td>1.62</td>
<td>2.00</td>
</tr>
<tr>
<td>∞</td>
<td>-2.58</td>
<td>-2.23</td>
<td>-1.95</td>
<td>-1.62</td>
<td>.89</td>
<td>1.28</td>
<td>1.62</td>
<td>1.99</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n</th>
<th>τ_μ</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>-3.75</td>
</tr>
<tr>
<td>50</td>
<td>-3.59</td>
</tr>
<tr>
<td>100</td>
<td>-3.50</td>
</tr>
<tr>
<td>250</td>
<td>-3.46</td>
</tr>
<tr>
<td>500</td>
<td>-3.44</td>
</tr>
<tr>
<td>750</td>
<td>-3.43</td>
</tr>
<tr>
<td>∞</td>
<td>-3.42</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n</th>
<th>τ_τ</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>-4.38</td>
</tr>
<tr>
<td>50</td>
<td>-4.15</td>
</tr>
<tr>
<td>100</td>
<td>-4.04</td>
</tr>
<tr>
<td>250</td>
<td>-3.98</td>
</tr>
<tr>
<td>500</td>
<td>-3.97</td>
</tr>
<tr>
<td>750</td>
<td>-3.96</td>
</tr>
<tr>
<td>∞</td>
<td>-3.96</td>
</tr>
</tbody>
</table>
Recalling that the numerator of \( n(\hat{\beta} - 1) \) has the limit distribution of \( \frac{1}{2} (\chi^2 - 1) \) where \( \chi^2 \) is a chi-square random variable with one degree of freedom one might be surprised that the limit distribution of \( n(\hat{\beta} - 1) \) is skew to the left rather than the right. Skewness to the left is explained by the correlation of the numerator and denominator, which approaches .8165 as \( n \to \infty \).

One might ask if \( P(n(\hat{\beta} - 1) > 0) \) and \( P(\tau > 0) \) are zero since the 99\(^{th}\) percentiles of their distributions are negative. In each case the Monte Carlo studies yielded a few estimates that exceeded zero.

**Goodness of Fit**

Since there were several observed percentiles at some \( n \) values (for example, 4 replications at \( n = 25 \)) we were able to calculate lack of fit statistics for the regressions of the percentiles on \( n^{-1} \). Table 5.4 gives the F statistics for lack of fit. None exceeded the tabular .10 value of F for 5 and 7 degrees of freedom (4 and 7 for Model (5.2)) adding justification to the use of the regression estimated percentiles.

**Table 5.4** Lack of fit F statistics.

<table>
<thead>
<tr>
<th></th>
<th>.01</th>
<th>.025</th>
<th>.05</th>
<th>.10</th>
<th>.90</th>
<th>.95</th>
<th>.975</th>
<th>.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n(\hat{\beta} - 1) )</td>
<td>1.385</td>
<td>.720</td>
<td>.602</td>
<td>.368</td>
<td>1.114</td>
<td>.272</td>
<td>.111</td>
<td>.112</td>
</tr>
<tr>
<td>( n(\hat{\beta} - 1) )</td>
<td>.956</td>
<td>.427</td>
<td>.338</td>
<td>.302</td>
<td>.700</td>
<td>.329</td>
<td>.445</td>
<td>.778</td>
</tr>
<tr>
<td>( n(\hat{\beta}_\mu - 1) )</td>
<td>.191</td>
<td>.569</td>
<td>.512</td>
<td>.536</td>
<td>1.977</td>
<td>1.278</td>
<td>.850</td>
<td>.503</td>
</tr>
<tr>
<td>( \tau )</td>
<td>.621</td>
<td>.358</td>
<td>.142</td>
<td>.422</td>
<td>1.286</td>
<td>.649</td>
<td>.353</td>
<td>.318</td>
</tr>
<tr>
<td>( \tau_\mu )</td>
<td>.592</td>
<td>.226</td>
<td>.305</td>
<td>.378</td>
<td>1.124</td>
<td>.386</td>
<td>.525</td>
<td>.600</td>
</tr>
<tr>
<td>( \tau_\tau )</td>
<td>.447</td>
<td>.308</td>
<td>.263</td>
<td>.219</td>
<td>1.255</td>
<td>2.312</td>
<td>.880</td>
<td>.446</td>
</tr>
</tbody>
</table>
Standard Errors

To evaluate the accuracy of the estimated percentiles, distribution free confidence intervals were computed for \( n = 25 \) and for the limiting distribution. The computation of these intervals is described in David (1970 pg. 13). The widths of the 95% confidence intervals divided by 2(1.96) are reported in Table 5.5. These numbers might be taken as approximations to the standard errors of the limiting distributions of the percentiles because the limiting distribution of estimated percentiles is normal.

Table 5.5  Length of confidence interval divided by 3.92.

<table>
<thead>
<tr>
<th>n</th>
<th>( n(\hat{\beta}-1) )</th>
<th>( n(\hat{\beta}_{\mu}-1) )</th>
<th>( n(\hat{\beta}_{\tau}-1) )</th>
<th>( \tau )</th>
<th>( \tau_{\mu} )</th>
<th>( \tau_{\tau} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>25</td>
<td>( \infty )</td>
<td>25</td>
<td>( \infty )</td>
<td>25</td>
<td>( \infty )</td>
</tr>
<tr>
<td>.01</td>
<td>.090</td>
<td>.105</td>
<td>.083</td>
<td>.117</td>
<td>.068</td>
<td>.097</td>
</tr>
<tr>
<td>.025</td>
<td>.054</td>
<td>.075</td>
<td>.052</td>
<td>.066</td>
<td>.058</td>
<td>.060</td>
</tr>
<tr>
<td>.05</td>
<td>.041</td>
<td>.051</td>
<td>.047</td>
<td>.061</td>
<td>.043</td>
<td>.046</td>
</tr>
<tr>
<td>.10</td>
<td>.026</td>
<td>.032</td>
<td>.034</td>
<td>.042</td>
<td>.033</td>
<td>.032</td>
</tr>
<tr>
<td>.90</td>
<td>.006</td>
<td>.005</td>
<td>.011</td>
<td>.011</td>
<td>.016</td>
<td>.009</td>
</tr>
<tr>
<td>.95</td>
<td>.007</td>
<td>.007</td>
<td>.012</td>
<td>.012</td>
<td>.021</td>
<td>.012</td>
</tr>
<tr>
<td>.975</td>
<td>.011</td>
<td>.009</td>
<td>.020</td>
<td>.012</td>
<td>.026</td>
<td>.013</td>
</tr>
<tr>
<td>.99</td>
<td>.017</td>
<td>.015</td>
<td>.022</td>
<td>.017</td>
<td>.035</td>
<td>.019</td>
</tr>
</tbody>
</table>

Since 25 and \( \infty \) are the smallest and largest values of \( n \) for which Monte Carlo results were obtained, the largest standard error for any of the regression estimates in Tables 5.2 and 5.3 is no larger than the largest of the two entries in Table 5.5.
Medians

The following table gives the empirical medians of the distributions of the six statistics for series of length \( n = 25, 100, \) and \( \infty \).

Table 5.6 Medians of sample distributions of estimators.

<table>
<thead>
<tr>
<th>Sample size</th>
<th>( n(\hat{\phi} - 1) )</th>
<th>( n(\hat{\mu} - 1) )</th>
<th>( n(\hat{\tau} - 1) )</th>
<th>( \tau )</th>
<th>( \tau_{\mu} )</th>
<th>( \tau_{\tau} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>-.806 (.009)</td>
<td>-4.221 (.014)</td>
<td>-8.481 (.022)</td>
<td>-1.529 (.003)</td>
<td>-2.135 (.003)</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>-1.842 (.012)</td>
<td>-4.340 (.021)</td>
<td>-8.994 (.030)</td>
<td>-1.557 (.005)</td>
<td>-2.174 (.004)</td>
<td></td>
</tr>
<tr>
<td>( \infty )</td>
<td>-1.855 (.009)</td>
<td>-4.351 (.015)</td>
<td>-9.109 (.014)</td>
<td>-1.566 (.003)</td>
<td>-2.181 (.002)</td>
<td></td>
</tr>
</tbody>
</table>

Notes on Use of Tables

The tabulated distribution of \( \hat{\phi} \) has been shown to be the distribution of the maximum likelihood estimator of \( \rho \) in the model \( Y_t = \rho Y_{t-1} + e_t \) given \( \rho = 1 \). From the preceding description of the Monte Carlo study we note that the tabulated distribution of \( \hat{\mu} \) is the distribution of the maximum likelihood estimator of \( \rho \) in the model \( Y_t = \alpha + \rho Y_{t-1} + e_t \) under the conditions \( \rho = 1 \) and \( \alpha = 0 \). Similarly, the distribution of \( \hat{\tau} \) is the distribution of the maximum likelihood estimate of \( \rho \) in the model \( Y_t = \alpha + \beta t + \rho Y_{t-1} + e_t \) under the conditions \( \beta = 0 \) and \( \rho = 1 \).

For models in which the observations occur later in the realization (that is the first observation is not \( e_1 \)) the first observation \( Y_1 \) should be subtracted from all the observations before computing \( \hat{\phi} \). The number of observations available for analysis is decreased by one and