1. Prove the Cauchy inequality for an inner product in a Hilbert space. That is, for any \( f(\cdot), g(\cdot) \), show that
\[
| < f, g > | \leq ||f||_H \cdot ||g||_H.
\]

**Hint:** Prove the inequality for 2 cases: (1) \( ||f||_H = ||g||_H = 0 \); (2) At least one of \( ||f||_H, ||g||_H \) is not zero.

2. In class, we discussed the estimation of \( f(t) \) in the following model
\[
y_i = f(t_i) + \epsilon_i, \quad \epsilon_i \sim N(0, \sigma^2) \tag{1}
\]
using mixed model representation where \( 0 < t_1 < t_2 < \cdots < t_n < 1 \). Let us consider the case where \( m = 1 \) (quadratic smoothing spline). In this case, the \((i,j)\)th element of \( \Sigma \) is \( \min(t_i, t_j) \) and \( T = (1, 1, \ldots, 1)^T \). Implement the EM algorithm for ML estimation of \( f(t) \), \( n = 100, t_i = i/n \), and \( f(t) = \sin(2\pi t) \) and \( \sigma^2 = 1 \). Run a simulation study (with 200 runs) to compare the average of (200) estimated \( f(t) \) to its true function.